Mathematical Analysis of Tetrahedron

Application of HCR’s Inverse Cosine Formula & Theory of Polygon

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1. Introduction: We very well know that a tetrahedron is a solid having 4 triangular faces, 6 edges & 4 vertices. Three triangular faces meet together at each of its four vertices & each of its six edges is shared (common) by two triangular faces. A tetrahedron is specified by the lengths of three edges meeting at any of its four vertices & the values of (three) angles between them consecutively. But if all three angles between the consecutive edges meeting at any vertex are known then we can easily calculate the internal (dihedral) angles between the consecutive triangular faces meeting at the same vertex & the solid angle subtended at the same vertex by the given tetrahedron. (See figure 1 showing a tetrahedron PQRS) Here we are to determine the internal (dihedral) angles between the consecutive triangular faces meeting at any of four vertices & the solid angle subtended by the tetrahedron at the same vertex given three angles $\alpha, \beta & \gamma$ between the consecutive (lateral) edges meeting at the same vertex.

2. Analysis of tetrahedron given the angles $\alpha, \beta & \gamma$ between the consecutive lateral edges:

Consider any tetrahedron PQRS having angles $\alpha, \beta & \gamma$ between the consecutive lateral edges PQ, PR, & PS $(\forall \alpha \leq \beta \leq \gamma)$ meeting at the vertex (apex) P.

Internal (dihedral) angles $\theta_1, \theta_2 & \theta_3$ between the consecutive lateral triangular faces $\Delta PSQ, \Delta PSR, \Delta PQR & \Delta PQS$ and $\Delta PRQ & \Delta PRS$ respectively: We know that the interior angle of between two consecutive triangular faces is measured normal to their common edge. (See figure 1 above)

Now the interior angles $\theta_1, \theta_2 & \theta_3$ between consecutive (lateral) triangular faces of the tetrahedron PQRS meeting at the vertex P, are determined/calculated by using HCR’s Inverse Cosine Formula according to which if $x, y & z$ are the angles between consecutive (lateral) edges meeting at any of four vertices of a tetrahedron then the angle (opposite to $a$) between two consecutive lateral faces is given as follows

$$\theta = \cos^{-1}\left(\frac{\cos x - \cos y \cos z}{\sin y \sin z}\right)$$

Now, setting the corresponding values in the above equation, we get all three interior angles as follows

$$\theta_1 = \cos^{-1}\left(\frac{\cos \alpha - \cos \beta \cos \gamma}{\sin \beta \sin \gamma}\right)$$

$$\theta_2 = \cos^{-1}\left(\frac{\cos \beta - \cos \alpha \cos \gamma}{\sin \alpha \sin \gamma}\right)$$

$$\theta_3 = \cos^{-1}\left(\frac{\cos \gamma - \cos \alpha \cos \beta}{\sin \alpha \sin \beta}\right)$$
Solid angle subtended by the tetrahedron PQRS at the common vertex P: For ease of calculation of the solid angle subtended by tetrahedron PQRS at the common vertex P (figure 1 above), let’s cut three equal segments PA = PB = PC = \(d\) from the lateral edges PS, PQ & PR respectively. Now, join the points A, B & C by the straight lines to obtain \(\Delta ABC\) which exerts a solid angle equal to that subtended by the original tetrahedron PQRS at its vertex P. Thus we would calculate the solid angle subtended by \(\Delta ABC\) at the (common) vertex P by two methods 1) Analytic & 2) Graphical as given below.

1. Analytic method for calculation of solid angle:

Sides of \(\Delta ABC\): Let the sides of \(\Delta ABC\) be \(a\), \(b\) & \(c\) opposite to its angles \(A\), \(B\) & \(C\) respectively.

In isosceles \(\Delta PBC\)

\[
\sin \frac{\alpha}{2} = \frac{\sin \frac{\alpha}{2}}{d} \Rightarrow a = 2d \sin \frac{\alpha}{2}
\]

Similarly,

\[
b = 2d \sin \frac{\beta}{2} \quad \text{&} \quad c = 2d \sin \frac{\gamma}{2}
\]

Now from HCR’s Axiom-2, we know that the perpendicular drawn from any vertex of a tetrahedron always passes through circumscribed centre of the (plane) triangle (in this case \(\Delta ABC\)) obtained by joining the points on the lateral edges equidistant from the same vertex. (See the figure 2)

Hence, the circumscribed radius \(R\) of \(\Delta ABC\) having its sides \(a\), \(b\) & \(c\) (all known) is calculated as follows

\[
s = \frac{a + b + c}{2} = \frac{2d \sin \frac{\alpha}{2} + 2d \sin \frac{\beta}{2} + 2d \sin \frac{\gamma}{2}}{2} = d \left( \sin \frac{\alpha}{2} + \sin \frac{\beta}{2} + \sin \frac{\gamma}{2} \right)
\]

\[
\Rightarrow \text{Area of } \Delta ABC, \Delta = \sqrt{(s-a)(s-b)(s-c)}
\]

Now, by substituting all the corresponding values in the above expression, we get

\[
\Delta = d^2 \sqrt{\frac{\sin \frac{\alpha}{2} + \sin \frac{\beta}{2} + \sin \frac{\gamma}{2}}{2} \left( \sin \frac{\beta}{2} + \sin \frac{\gamma}{2} - \sin \frac{\alpha}{2} \right) \left( \sin \frac{\gamma}{2} + \sin \frac{\alpha}{2} - \sin \frac{\beta}{2} \right) \left( \sin \frac{\alpha}{2} + \sin \frac{\beta}{2} - \sin \frac{\gamma}{2} \right)}
\]

\[
= d^2 \sqrt{\frac{\sin \frac{\alpha}{2} + \sin \frac{\beta}{2} + \sin \frac{\gamma}{2}}{2} \left( \sin \frac{\beta}{2} + \sin \frac{\gamma}{2} - \sin \frac{\alpha}{2} \right) \left( \sin \frac{\gamma}{2} + \sin \frac{\alpha}{2} - \sin \frac{\beta}{2} \right) \left( \sin \frac{\alpha}{2} + \sin \frac{\beta}{2} - \sin \frac{\gamma}{2} \right)}
\]

\[
= d^2 \sqrt{4 \sin^2 \frac{\alpha}{2} \sin^2 \frac{\beta}{2} - \left( \sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} - \sin^2 \frac{\gamma}{2} \right)^2}
\]
“Mathematical Analysis of Tetrahedron (dihedral angles between the consecutive faces & solid angle subtended by the tetrahedron at its vertex given the angles between consecutive edges meeting at the same vertex)”

Hence, the circumscribed radius \( R \) of \( \Delta ABC \) having its sides \( a, b \) & \( c \) (all known) is given as follows

\[
R = \frac{abc}{4A} = \frac{2\sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}}{\sqrt{4\sin^2 \frac{\alpha}{2} \sin^2 \frac{\beta}{2} - \left( \sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} - \sin^2 \frac{\gamma}{2} \right)^2}}
\]

\[
= \frac{2\sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}}{\sqrt{4\sin^2 \frac{\alpha}{2} \sin^2 \frac{\beta}{2} - \left( \sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} - \sin^2 \frac{\gamma}{2} \right)^2}} = Kd \text{ (let’s assume)}
\]

\[ \therefore R = Kd \text{ where, } K = \frac{2\sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}}{\sqrt{4\sin^2 \frac{\alpha}{2} \sin^2 \frac{\beta}{2} - \left( \sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} - \sin^2 \frac{\gamma}{2} \right)^2}} = \text{ constant } \quad (0 < K < 1) \]

\[ \forall (\alpha + \beta) > \gamma, (\beta + \gamma) > \alpha, (\gamma + \alpha) > \beta \quad \& \quad \alpha + \beta + \gamma < 360^\circ \]

Note: For ease of understanding & calculations always follow the sequence \( \alpha \leq \beta \leq \gamma \) for given (known) values of angles \( \alpha \), \( \beta \) & \( \gamma \) between the consecutive lateral edges meeting at a vertex of any tetrahedron.

Hence, the normal height \( h \) of \( \Delta ABC \) from the vertex (apex) \( P \) of the tetrahedron \( PQRS \) is given as follows

In right \( \Delta P0A \)

\[ PO = \sqrt{(PA)^2 - (OA)^2} = \sqrt{d^2 - R^2} = \sqrt{d^2 - (Kd)^2} = d\sqrt{1 - K^2} \]

\[ \therefore h = d\sqrt{1 - K^2} \]

Now, in right \( \Delta OMB \)

\[ OM = \sqrt{(BO)^2 - (MB)^2} = \sqrt{R^2 - \left(\frac{a}{2}\right)^2} = \sqrt{(Kd)^2 - \left(\frac{2\sin \frac{\alpha}{2}}{2}\right)^2} = d\sqrt{K^2 - \sin^2 \frac{\alpha}{2}} \]

\[ \therefore OM = d\sqrt{K^2 - \sin^2 \frac{\alpha}{2}} \]

Now, from HCR’s Theory of Polygon, the solid angle subtended by the right triangle having its orthogonal sides \( \alpha \& \beta \) at any point lying at a height \( h \) on the vertical axis passing through the vertex common to the side \( \alpha \& \beta \) is given from standard formula as

\[ \omega = \sin^{-1} \left( \frac{b}{\sqrt{b^2 + a^2}} \right) - \sin^{-1} \left( \frac{b}{\sqrt{b^2 + a^2}} \left( \frac{h}{\sqrt{h^2 + a^2}} \right) \right) \]

Hence, the solid angle \( \omega_{\Delta OBC} \) subtended by the isosceles \( \Delta OBC \) at the vertex \( P \) of the tetrahedron

\[ = \omega_{\Delta OMB} + \omega_{\Delta OMC} = 2(\omega_{\Delta OMB}) = 2(\text{solid angle subtended by the right } \Delta OMB) \]

\[ \Rightarrow \omega_{\Delta OBC} = 2 \left[ \sin^{-1} \left( \frac{(MB)}{\sqrt{(MB)^2 + (OM)^2}} \right) - \sin^{-1} \left( \frac{(MB)}{\sqrt{(MB)^2 + (OM)^2}} \left( \frac{(PO)}{\sqrt{(PO)^2 + (OM)^2}} \right) \right) \right] \]
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Hence, by setting the corresponding values in the above formula, we get

\[ \omega_{\Delta OBC} = 2 \left[ \sin^{-1} \left( \frac{a}{\sqrt{(a/2)^2 + (OM)^2}} \right) - \sin^{-1} \left( \frac{h}{\sqrt{(h/2)^2 + (OM)^2}} \right) \right] \]

\[ = 2 \sin^{-1} \left( \frac{2 \sin \frac{\alpha}{2}}{2} \right) \]

\[ - \sin^{-1} \left( \frac{2 \sin \frac{\alpha}{2}}{2} \right) \]

\[ = 2 \sin^{-1} \left( \frac{dsin \frac{\alpha}{2}}{d^2 \sin \frac{\alpha}{2} + K^2 d^2 - d^2 \sin \frac{\alpha}{2}} \right) \]

\[ - \sin^{-1} \left( \frac{1}{Kd} \right) \]

\[ = 2 \left[ \sin^{-1} \left( \frac{\sin \frac{\alpha}{2}}{K} \right) - \sin^{-1} \left( \tan \frac{\alpha}{2} \sqrt{\frac{1}{K^2} - 1} \right) \right] = \omega_1 \text{ (let)} \]

Similarly, we have

\[ \omega_{\Delta OAC} = 2 \sin^{-1} \left( \frac{\sin \frac{\beta}{2}}{K} \right) - \sin^{-1} \left( \tan \frac{\beta}{2} \sqrt{\frac{1}{K^2} - 1} \right) = \omega_2 \text{ (let)} \]
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\[
\omega_{\Delta OAB} = 2 \left( \sin^{-1} \left( \frac{\sin \frac{\gamma}{2}}{K} \right) - \sin^{-1} \left( \tan \frac{\gamma}{2} \sqrt{\left( \frac{1}{K} \right)^2 - 1} \right) \right) = \omega_3 \ (let)
\]

Now, we must check out the nature of \( \Delta ABC \) whether it is an acute, a right or an obtuse triangle. Let’s assume the largest angle is \( \gamma \) among known values of \( \alpha, \beta \) and \( \gamma \) hence we can determine the largest angle \( C \) of \( \Delta ABC \) using cosine formula as follows

\[
\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{(2\sin \frac{\alpha}{Z})^2 + (2\sin \frac{\beta}{Z})^2 - (2\sin \frac{\gamma}{Z})^2}{2\left(2\sin \frac{\alpha}{Z}\right)\left(2\sin \frac{\beta}{Z}\right)} = \frac{\sin^2 \frac{\alpha}{Z} + \sin^2 \frac{\beta}{Z} - \sin^2 \frac{\gamma}{Z}}{2\sin \frac{\alpha}{Z}\sin \frac{\beta}{Z}}
\]

\[
\therefore \cos C = \frac{\sin^2 \frac{\alpha}{Z} + \sin^2 \frac{\beta}{Z} - \sin^2 \frac{\gamma}{Z}}{2\sin \frac{\alpha}{Z}\sin \frac{\beta}{Z}} \text{ or } C = \cos^{-1} \left( \frac{\sin^2 \frac{\alpha}{Z} + \sin^2 \frac{\beta}{Z} - \sin^2 \frac{\gamma}{Z}}{2\sin \frac{\alpha}{Z}\sin \frac{\beta}{Z}} \right) \quad \forall \gamma \geq \beta \geq \alpha
\]

By substituting the known values of angles \( \alpha, \beta \) and \( \gamma \) in the above expression, we can directly calculate the value of the largest angle \( C \) to check out the nature of \( \Delta ABC \). Thus, there arise two cases to calculate the solid angle subtended by the plane \( \Delta ABC \) at the vertex \( P \) of tetrahedron \( PQRS \) as follows

**Case 1:** \( \Delta ABC \) is an acute or a right triangle (\( \forall \gamma \geq \beta \geq \alpha \) & \( C \leq 90^\circ \)):

In this case, the foot point \( O \) of the perpendicular drawn from the vertex \( P \) to the plane of acute \( \Delta ABC \) lies within or on (in case of right triangle) the boundary of this triangle. (See the figure 2 above) All the values of solid angles \( \omega_1, \omega_2 \) & \( \omega_3 \) corresponding to the angles \( \alpha, \beta \) and \( \gamma \) respectively of a tetrahedron are taken as positive. Hence, the solid angle \( (\omega) \) subtended by the tetrahedron \( PQRS \) at the vertex \( P \) is equal to the solid angle \( (\omega_{\Delta ABC}) \) subtended by the acute/right \( \Delta ABC \) at the vertex \( P \) of tetrahedron which is given as the sum of magnitudes of solid angles as follows

\[
\omega = \omega_{\Delta ABC} = \omega_{\Delta OBC} + \omega_{\Delta OAC} + \omega_{\Delta OAB} = \omega_1 + \omega_2 + \omega_3
\]

**Case 2:** \( \Delta ABC \) is an obtuse triangle (\( \forall \gamma > \beta \geq \alpha \) & \( C > 90^\circ \)):

In this case, the foot point \( O \) of the perpendicular drawn from the vertex \( P \) to the plane of obtuse \( \Delta ABC \) lies outside the boundary of this triangle. (See the figure 3 above). In this case, solid angles \( \omega_1 \) & \( \omega_2 \) corresponding to the angles \( \alpha \) and \( \beta \) respectively are taken as positive while solid angle \( \omega_3 \) corresponding to the largest angle \( \gamma \) of a tetrahedron is taken as negative. Hence, the solid angle subtended by the tetrahedron \( PQRS \) at the vertex \( P \) is equal to the solid angle \( (\omega_{\Delta ABC}) \) subtended by the obtuse \( \Delta ABC \) at the vertex \( P \) of tetrahedron which is given as the **algebraic sum of solid angles** as follows

\[
\omega = \omega_{\Delta ABC} = \omega_{\Delta OBC} + \omega_{\Delta OAC} - \omega_{\Delta OAB} = \omega_1 + \omega_2 - \omega_3
\]

**Application of “HCR’s Theory of Polygon” proposed by H. C. Rajpoot (2014)**

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2. Graphical method for calculation of solid angle:

In this method, we first plot the diagram of \(\Delta ABC\) having known sides \(a, b, \text{&} c\) by taking a suitable multiplying factor \(d\) (as mentioned above) \& then specify the location of foot of perpendicular (F.O.P.) i.e. the circumscribed centre of \(\Delta ABC\) then draw the perpendiculars from circumscribed centre \(O\) to all the opposite sides to divide it (i.e. \(\Delta ABC\)) into elementary right triangles \& use standard formula-1 of right triangle for calculating the solid angle subtended by each of the elementary right triangles at the centre of sphere which is given as follows

\[
\omega = \sin^{-1} \left( \frac{b}{\sqrt{b^2 + a^2}} \right) - \sin^{-1} \left( \frac{b}{\sqrt{b^2 + a^2}} \right) \left( \frac{h}{\sqrt{h^2 + a^2}} \right)
\]

Then find out the algebraic sum (\(\omega\)) of the solid angles subtended by the elementary right triangles at the vertex of the given tetrahedron depending on the nature of the triangle \(ABC\).

\[\therefore \text{Solid angle subtended by the tetrahedron at the vertex} = \omega = \text{algebraic sum of } \omega_1, \omega_2 \text{& } \omega_3\]

**Important deductions: 1.** Consider any of eight octants in 3-D co-ordinate system, if three co-ordinates axes \(X, Y \text{&} Z\) represent the consecutive edges meeting at the origin (vertex) of a tetrahedron then in this case the planes \(XY, YZ \text{&} ZX\) will represent the consecutive lateral faces meeting/intersecting at the origin (vertex) \& we have

\[\alpha = \beta = \gamma = 90^\circ \text{ (angle between any two orthogonal axes in 3D system)}\]

Now, all the interior angles \(\theta_1, \theta_2 \text{&} \theta_3\) opposite to the angles \(\alpha, \beta \text{&} \gamma\) respectively between the consecutive lateral faces i.e. planes \(XY, YZ \text{&} ZX\) of the tetrahedron can be easily calculated by using inverse cosine formula as follows

\[
\theta_1 = \cos^{-1} \left( \frac{\cos\alpha - \cos\beta \cos\gamma}{\sin\beta \sin\gamma} \right) = \cos^{-1} \left( \frac{\cos\alpha - \cos\beta \cos\gamma}{\sin\beta \sin\gamma} \right) = 90^\circ
\]

\[
\theta_2 = \cos^{-1} \left( \frac{\cos\beta - \cos\gamma \cos\alpha}{\sin\gamma \sin\alpha} \right) = \cos^{-1} \left( \frac{\cos\beta - \cos\gamma \cos\alpha}{\sin\gamma \sin\alpha} \right) = 90^\circ
\]

\[
\theta_3 = \cos^{-1} \left( \frac{\cos\gamma - \cos\alpha \cos\beta}{\sin\alpha \sin\beta} \right) = \cos^{-1} \left( \frac{\cos\gamma - \cos\alpha \cos\beta}{\sin\alpha \sin\beta} \right) = 90^\circ
\]

The above values show that the angle between any two orthogonal planes in 3-D system is \(90^\circ\).

Now, calculate the constant \(K\) by using the formula as follows

\[
K = \frac{2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}}{\sqrt{4 \sin^2 \frac{\alpha}{2} \sin^2 \frac{\beta}{2} - \left( \sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} - \sin^2 \frac{\gamma}{2} \right)^2}}
\]

\[
K = \frac{2 \sin \frac{90^\circ}{2} \sin \frac{90^\circ}{2} \sin \frac{90^\circ}{2}}{\sqrt{4 \sin^2 \frac{90^\circ}{2} \sin^2 \frac{90^\circ}{2} - \left( \sin^2 \frac{90^\circ}{2} + \sin^2 \frac{90^\circ}{2} - \sin^2 \frac{90^\circ}{2} \right)^2}} = \frac{\left( \sqrt{2} \right)}{\sqrt{3}} = \sqrt{\frac{2}{3}} = \sqrt{\frac{2}{3}}
\]

Now, by substituting all the corresponding values, we get
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\[ \omega_1 = 2 \left[ \sin^{-1} \left( \frac{\sin \frac{\alpha}{2}}{K} \right) - \sin^{-1} \left( \tan \frac{\alpha}{2} \sqrt{\frac{1}{K} - 1} \right) \right] \]

\[ = 2 \left[ \sin^{-1} \left( \frac{\sin 90^\circ}{2} \right) - \sin^{-1} \left( \frac{90^\circ}{2} \sqrt{\frac{1}{2} - 1} \right) \right] \]

\[ \omega_1 = 2 \left[ \sin^{-1} \left( \frac{\sqrt{3}}{2} \right) - \sin^{-1} \left( \frac{1}{\sqrt{2}} \right) \right] = 2 \left[ \frac{\pi}{3} - \frac{\pi}{4} \right] = \frac{\pi}{6} \]

Similarly, we can find out the following values

\[ \Rightarrow \omega_2 = \frac{\pi}{6} \text{ and } \omega_3 = \frac{\pi}{6} \]

The largest angle of \( \triangle ABC \) is \( C \) which is calculated by using cosine formula as follows

\[ C = \cos^{-1} \left( \frac{\sin^2 \alpha + \sin^2 \beta - \sin^2 \gamma}{2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2}} \right) = \cos^{-1} \left( \frac{\sin^2 90^\circ + \sin^2 90^\circ - \sin^2 90^\circ}{2 \sin \frac{90^\circ}{2} \sin \frac{90^\circ}{2}} \right) = \cos^{-1} \left( \frac{1}{2} \right) = 60^\circ \]

\[ \Rightarrow C = 60^\circ < 90^\circ \]

Hence, the plane \( \triangle ABC \) is an acute angled triangle.

Hence the foot of perpendicular (F.O.P.) drawn from the vertex (origin) to the plane of \( \triangle ABC \) will lie within the boundary of \( \triangle ABC \) (See the figure 2 above) hence, the solid angle subtended by the tetrahedron at the vertex is the sum of the magnitudes of solid angles as follows

\[ \omega = \omega_1 + \omega_2 + \omega_3 = \frac{\pi}{6} + \frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{2} \text{ sr} \quad (\text{solid angle subtended by each octant at the origin}) \]

Above value shows that the solid angle subtended by each of eight octants in 3-D co-ordinate system at the origin is \( \frac{\pi}{2} \text{ sr} \). It can also be calculated by the following expression

\[ \text{solid angle subtended by each octant at the origin} = \frac{\text{total solid angle}}{\text{no. of octants}} = \frac{4\pi}{8} = \frac{\pi}{2} \text{ sr} \]

2. Consider a regular tetrahedron which has four congruent equilateral triangular faces each three meeting at each of four identical vertices. Hence, for any vertex of a regular tetrahedron, we have

\[ \alpha = \beta = \gamma = 60^\circ \quad (\text{angle between any two consecutive edges of a regular tetrahedron}) \]

Now, all the interior angles \( \theta_1, \theta_2 \) & \( \theta_3 \) opposite to the angles \( \alpha, \beta \) & \( \gamma \) respectively between the consecutive equilateral triangular faces of the regular tetrahedron can be easily calculated by using inverse cosine formula as follows

\[ \Rightarrow \theta_1 = \cos^{-1} \left( \frac{\cos \alpha - \cos \beta \cos \gamma}{\sin \beta \sin \gamma} \right) = \cos^{-1} \left( \frac{\cos 60^\circ - \cos 60^\circ \cos 60^\circ}{\sin 60^\circ \sin 60^\circ} \right) = \cos^{-1} \left( \frac{1}{3} \right) \approx \text{70.52877937}^\circ \]

\[ \theta_2 = \cos^{-1} \left( \frac{\cos \beta - \cos \alpha \cos \gamma}{\sin \alpha \sin \gamma} \right) = \cos^{-1} \left( \frac{\cos 60^\circ - \cos 60^\circ \cos 60^\circ}{\sin 60^\circ \sin 60^\circ} \right) = \cos^{-1} \left( \frac{1}{3} \right) \approx \text{70.52877937}^\circ \]
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\[ \theta_3 = \cos^{-1} \left( \frac{\cos \gamma - \cos \alpha \cos \beta}{\sin \alpha \sin \beta} \right) = \cos^{-1} \left( \frac{\cos 60^\circ - \cos 60^\circ \cos 60^\circ}{\sin 60^\circ \sin 60^\circ} \right) = \cos^{-1} \left( \frac{1}{3} \right) \approx 70.52877937^\circ \]

The above values show that the dihedral angle between any two consecutive equilateral triangular faces of a regular tetrahedron is \( \cos^{-1}(1/3) \approx 70.52877937^\circ \)

Now, calculate the constant \( K \) by using the formula as follows

\[
K = \frac{2 \sin \frac{60^\circ}{2} \sin \frac{60^\circ}{2} \sin \frac{60^\circ}{2}}{\sqrt{4 \sin^2 \frac{60^\circ}{2} \sin^2 \frac{60^\circ}{2} \sin^2 \frac{60^\circ}{2} - \left( \sin^2 \frac{60^\circ}{2} + \sin^2 \frac{60^\circ}{2} - \sin^2 \frac{60^\circ}{2} \right)^2}} = \frac{(1/4)}{\sqrt{3}} = \frac{1}{\sqrt{3}}
\]

Now, by substituting all the corresponding values, we get

\[
\Rightarrow \omega_1 = 2 \left[ \sin^{-1} \left( \frac{\sin \frac{60^\circ}{2}}{K} \right) - \sin^{-1} \left( \tan \frac{\alpha}{2} \sqrt{\frac{1}{K^2} - 1} \right) \right]
\]

\[
= 2 \left[ \sin^{-1} \left( \frac{\sin 60^\circ}{2} \left(\frac{1}{\sqrt{3}}\right) \right) - \sin^{-1} \left( \tan \frac{60^\circ}{2} \sqrt{\frac{1}{3} - 1} \right) \right] = 2 \left[ \frac{\pi}{3} - \sin^{-1} \left( \frac{\sqrt{3}}{2} \right) \right]
\]

Similarly, we can find out the following values

\[
\Rightarrow \omega_2 = 2 \left[ \frac{\pi}{3} - \sin^{-1} \left( \frac{\sqrt{3}}{2} \right) \right] \quad \& \quad \omega_3 = 2 \left[ \frac{\pi}{3} - \sin^{-1} \left( \frac{\sqrt{3}}{2} \right) \right]
\]

The largest angle of \( \Delta ABC \) is \( C \) which is calculated by using cosine formula as follows

\[
C = \cos^{-1} \left( \frac{\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} - \sin^2 \frac{\gamma}{2}}{2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2}} \right) = \cos^{-1} \left( \frac{\sin^2 60^\circ + \sin^2 60^\circ - \sin^2 60^\circ}{2 \sin 60^\circ \sin 60^\circ} \right) = \cos^{-1} \left( \frac{1}{2} \right) = 60^\circ
\]

\[
\Rightarrow C = 60^\circ < 90^\circ
\]

Hence, the plane \( \Delta ABC \) is an acute angled triangle.

Hence the foot of perpendicular (F.O.P.) drawn from the vertex of tetrahedron to the plane of \( \Delta ABC \) will lie within the boundary of \( \Delta ABC \) (See the figure 2 above) hence, the solid angle subtended by the tetrahedron at the vertex is the sum of the magnitudes of solid angles as follows
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\[ \omega = \omega_1 + \omega_2 + \omega_3 = 2 \left[ \frac{\pi}{3} - \sin^{-1}\left( \frac{2}{\sqrt{3}} \right) \right] + 2 \left[ \frac{\pi}{3} - \sin^{-1}\left( \frac{2}{\sqrt{3}} \right) \right] + 2 \left[ \frac{\pi}{3} - \sin^{-1}\left( \frac{2}{\sqrt{3}} \right) \right] \]

\[ = 2\pi - 6 \sin^{-1}\left( \frac{2}{\sqrt{3}} \right) = 0.551285598 \text{ sr} \]

Above value shows that the solid angle subtended by a regular tetrahedron at any of its four vertices is \( 0.551285598 \text{ sr} \). It can also be calculated by HCR’s standard formula of solid angle as follows

\[ \omega = 2\pi - 2\sin^{-1}\left( \cos \frac{\pi}{n} \sqrt{\tan^2 \frac{\pi}{n} - \tan^2 \frac{\alpha}{2}} \right) \]

For a regular tetrahedron, we have

\[ n = \text{no. of sides in each face} = 3 \quad \& \quad \alpha = \text{angle between lateral edges} = 60^\circ \]

\[ \therefore \omega = 2\pi - 2(3)\sin^{-1}\left( \cos \frac{\pi}{3} \sqrt{\tan^2 \frac{\pi}{3} - \tan^2 \frac{60^\circ}{2}} \right) = 2\pi - 6 \sin^{-1}\left( \frac{1}{2} \sqrt{\left(\sqrt{3}\right)^2 - \left(\frac{1}{\sqrt{3}}\right)^2} \right) \]

\[ = 2\pi - 6 \sin^{-1}\left( \frac{1}{2} \sqrt{\frac{8}{3}} \right) = 2\pi - 6 \sin^{-1}\left( \frac{2}{3} \right) \approx 0.551285598 \text{ sr} \]

Both the above results are equal hence the generalized formula of a tetrahedron is verified.

**Illustrative Numerical Examples**

These examples are based on all above articles which are very practical and directly & simply applicable to calculate the dihedral angles between the consecutive faces & the solid angle subtended by the tetrahedron at the vertex. For ease of understanding & calculations, value of angle \( \gamma \) of \( \triangle ABC \) is taken as the largest one).

**Example 1**: Calculate the interior angles between the consecutive faces & the solid angle subtended by a tetrahedron at the vertex such that the angles between the consecutive edges meeting at the same vertex are \( 30^\circ, 40^\circ \& 50^\circ \).

**Sol.** Here, we have

\[ \alpha = 30^\circ, \beta = 40^\circ \& \gamma = 50^\circ \Rightarrow \theta_1, \theta_2 \& \theta_3 = ? \& \text{solid angle, } \omega = ? \]

Now, all the interior angles \( \theta_1, \theta_2 \& \theta_3 \) opposite to the angles \( \alpha, \beta \& \gamma \) respectively of the tetrahedron can be easily calculated by using inverse cosine formula as follows

\[\Rightarrow \theta_1 = \cos^{-1}\left( \frac{\cos \alpha - \cos \beta \cos \gamma}{\sin \beta \sin \gamma} \right) = \cos^{-1}\left( \frac{\cos 30^\circ - \cos 40^\circ \cos 50^\circ}{\sin 40^\circ \sin 50^\circ} \right) \approx 40.64407403^\circ \]

\[ \approx 40^\circ 38' 38.67'' \]

\[ \theta_2 = \cos^{-1}\left( \frac{\cos \beta - \cos \gamma \cos \alpha}{\sin \beta \sin \alpha} \right) = \cos^{-1}\left( \frac{\cos 40^\circ - \cos 50^\circ \cos 30^\circ}{\sin 50^\circ \sin 30^\circ} \right) \approx 56.86341165^\circ \approx 56^\circ 51' 48.28'' \]
Now, calculate the constant K by using the formula as follows

\[ K = \frac{2\sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}}{\sqrt{4\sin^2 \frac{\alpha}{2} \sin^2 \frac{\beta}{2} - (\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} - \sin^2 \frac{\gamma}{2})^2}} \]

Now, by substituting all the corresponding values, we get

\[ K = \frac{2\sin 30^\circ \sin 40^\circ \sin 50^\circ}{\sqrt{4\sin^2 30^\circ \sin^2 40^\circ - (\sin^2 30^\circ + \sin^2 40^\circ - \sin^2 50^\circ)^2}} \approx 0.422811997 \]

\[ \Rightarrow \omega_1 = 2 \left[ \sin^{-1} \left( \frac{\sin \frac{\alpha}{2}}{K} \right) - \sin^{-1} \left( \frac{\tan \frac{\alpha}{2} \sqrt{\frac{1}{K} - 1}}{} \right) \right] \approx 0.094028018 \text{ sr} \]

\[ \Rightarrow \omega_2 = 2 \left[ \sin^{-1} \left( \frac{\sin \frac{\beta}{2}}{K} \right) - \sin^{-1} \left( \frac{\tan \frac{\beta}{2} \sqrt{\frac{1}{K} - 1}}{} \right) \right] \approx 0.094962792 \text{ sr} \]

\[ \Rightarrow \omega_3 = 2 \left[ \sin^{-1} \left( \frac{\sin \frac{\gamma}{2}}{K} \right) - \sin^{-1} \left( \frac{\tan \frac{\gamma}{2} \sqrt{\frac{1}{K} - 1}}{} \right) \right] \approx 0.00626143758 \text{ sr} \]

The largest angle of \( \Delta ABC \) is \( C \) which is calculated by using cosine formula as follows

\[ \cos^{-1} \left( \frac{\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} - \sin^2 \frac{\gamma}{2}}{2\sin \frac{\alpha}{2} \sin \frac{\beta}{2}} \right) = \cos^{-1} \left( \frac{\sin^2 30^\circ + \sin^2 40^\circ - \sin^2 50^\circ}{2\sin 30^\circ \sin 40^\circ} \right) \approx 88.26545646^\circ < 90^\circ \]

Hence, the plane \( \Delta ABC \) is an acute angled triangle.

Hence the foot of perpendicular (F.O.P.) drawn from the vertex of tetrahedron to the plane of \( \Delta ABC \) will lie within the boundary of \( \Delta ABC \) (See the figure 2 above) hence, the solid angle subtended by the tetrahedron at the vertex is the sum of the magnitudes of solid angles as follows

\[ \omega = \omega_1 + \omega_2 + \omega_3 \approx 0.094028018 + 0.094962792 + 0.00626143758 \approx 0.195252247 \text{ sr} \quad \text{Ans} \]
The above value of area implies that the given tetrahedron subtends a solid angle $\approx 0.195252247 \text{ sr}$ at the vertex irrespective of its geometrical dimensions.

**Example 2:** Calculate the interior angles between the consecutive faces & the solid angle subtended by a tetrahedron at the vertex such that the angles between the consecutive edges meeting at the same vertex are $40^\circ, 70^\circ \& 85^\circ$.

**Sol.** Here, we have

$$\alpha = 40^\circ, \beta = 70^\circ \& \gamma = 85^\circ \Rightarrow \theta_1, \theta_2 \& \theta_3 = ? \& \text{solid angle, } \omega = ?$$

Now, all the interior angles $\theta_1, \theta_2 \& \theta_3$ opposite to the angles $\alpha, \beta \& \gamma$ respectively of the tetrahedron can be easily calculated by using inverse cosine formula as follows

$$\Rightarrow \theta_1 = \cos^{-1}\left(\frac{\cos \alpha - \cos \beta \cos \gamma}{\sin \beta \sin \gamma}\right) = \cos^{-1}\left(\frac{\cos 40^\circ - \cos 70^\circ \cos 85^\circ}{\sin 70^\circ \sin 85^\circ}\right) \approx 38.1424026^\circ \approx 38^\circ 8'32.65''$$

$$\theta_2 = \cos^{-1}\left(\frac{\cos \beta - \cos \alpha \cos \gamma}{\sin \alpha \sin \gamma}\right) = \cos^{-1}\left(\frac{\cos 70^\circ - \cos 40^\circ \cos 85^\circ}{\sin 40^\circ \sin 85^\circ}\right) \approx 64.5415945^\circ \approx 64^\circ 32'29.58''$$

$$\theta_3 = \cos^{-1}\left(\frac{\cos \gamma - \cos \alpha \cos \beta}{\sin \alpha \sin \beta}\right) = \cos^{-1}\left(\frac{\cos 85^\circ - \cos 40^\circ \cos 70^\circ}{\sin 40^\circ \sin 70^\circ}\right) \approx 106.826295^\circ$$

Now, calculate the constant $K$ by using the formula as follows

$$K = \frac{2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}}{\sqrt{4 \sin^2 \frac{\alpha}{2} \sin^2 \frac{\beta}{2} - \left(\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} - \sin^2 \frac{\gamma}{2}\right)^2}}$$

Now, by substituting all the corresponding values, we get

$$K = \frac{2 \sin \frac{40^\circ}{2} \sin \frac{70^\circ}{2} \sin \frac{85^\circ}{2}}{\sqrt{4 \sin^2 \frac{40^\circ}{2} \sin^2 \frac{70^\circ}{2} - \left(\sin^2 \frac{40^\circ}{2} + \sin^2 \frac{70^\circ}{2} - \sin^2 \frac{85^\circ}{2}\right)^2}} \approx 0.675830167$$

$$\Rightarrow \omega_1 = 2 \left[ \sin^{-1}\left(\frac{\sin \frac{\alpha}{2}}{K}\right) - \sin^{-1}\left(\tan \frac{\alpha}{2} \sqrt{\left(\frac{1}{K}\right)^2 - 1}\right)\right]$$

$$\therefore \omega_1 = 2 \left[ \sin^{-1}\left(\frac{\sin \frac{40^\circ}{2}}{0.675830167}\right) - \sin^{-1}\left(\tan \frac{40^\circ}{2} \sqrt{\left(\frac{1}{0.675830167}\right)^2 - 1}\right)\right] \approx 0.244883226 \text{ sr}$$

$$\Rightarrow \omega_2 = 2 \left[ \sin^{-1}\left(\frac{\sin \frac{\beta}{2}}{K}\right) - \sin^{-1}\left(\tan \frac{\beta}{2} \sqrt{\left(\frac{1}{K}\right)^2 - 1}\right)\right]$$

$$\therefore \omega_2 = 2 \left[ \sin^{-1}\left(\frac{\sin \frac{70^\circ}{2}}{0.675830167}\right) - \sin^{-1}\left(\tan \frac{70^\circ}{2} \sqrt{\left(\frac{1}{0.675830167}\right)^2 - 1}\right)\right] \approx 0.289166683 \text{ sr}$$
Mathematical Analysis of Tetrahedron (dihedral angles between the consecutive faces & solid angle subtended by the tetrahedron at its vertex given the angles between consecutive edges meeting at the same vertex)

\[ \omega_3 = 2 \left[ \sin^{-1} \left( \frac{\sin \frac{\gamma}{2}}{K} \right) - \sin^{-1} \left( \tan \frac{\gamma}{2} \sqrt{\frac{1}{K} - 1} \right) \right] \]

\[ \therefore \omega_3 = 2 \left[ \sin^{-1} \left( \frac{\sin 85^\circ}{0.675830167} \right) - \sin^{-1} \left( \tan \frac{85^\circ}{2} \sqrt{\frac{1}{0.675830167} - 1} \right) \right] \approx 0.018999357 \text{ sr} \]

The largest angle of \( \Delta ABC \) is \( C \) which is calculated by using cosine formula as follows

\[ C = \cos^{-1} \left( \frac{\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} - \sin^2 \frac{\gamma}{2}}{2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2}} \right) = \cos^{-1} \left( \frac{\sin^2 \frac{40^\circ}{2} + \sin^2 \frac{70^\circ}{2} - \sin^2 \frac{85^\circ}{2}}{2 \sin \frac{40^\circ}{2} \sin \frac{70^\circ}{2}} \right) \approx 91.52686653^\circ > 90^\circ \]

Hence, the plane \( \Delta ABC \) is an obtuse angled triangle.

Hence the foot of perpendicular (F.O.P.) drawn from the vertex of tetrahedron to the plane of \( \Delta ABC \) will lie outside the boundary of plane \( \Delta ABC \) (See the figure 3 above) hence, the solid angle subtended by the tetrahedron at the vertex is the algebraic sum of the solid angles as follows

\[ \omega = \omega_1 + \omega_2 - \omega_3 \approx 0.244883226 + 0.289166683 - 0.018999357 \approx 0.515050552 \text{ sr} \quad \text{Ans} \]

The above value of area implies that the given tetrahedron subtends a solid angle \( \approx 0.515050552 \text{ sr} \) at the vertex irrespective of its geometrical dimensions.

Conclusion: All the articles above have been derived by Mr H.C. Rajpoot by using simple geometry & trigonometry. All above articles (formula) are very practical & simple to apply in case of any tetrahedron to calculate the internal (dihedral) angles between the consecutive lateral faces meeting at any of four vertices & the solid angle subtended by it (tetrahedron) at the vertex when the angles between the consecutive edges meeting at the same vertex are known. These are the generalised formula which can also be applied in case of three faces meeting at the vertex of various regular & uniform polyhedrons to calculate the solid angle subtended by the polyhedron at its vertex.

Note: Above articles had been derived & illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)

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