Mathematical analysis of identical circles touching one another on the whole spherical surface

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Introduction: Here, we are to analyse the identical circles, each having a flat radius \( r \), touching one another on a whole spherical surface with a certain radius \( R \) by finding out

1. Relation between \( R \) & \( r \)
2. Radius of each circle as a great circle arc (\( \text{arc} \) \( r \)) &
3. Total surface area & its percentage (%) covered by all the circles on the whole spherical surface.

For such cases, each of the identical circles is assumed to be inscribed by each of the congruent regular polygonal (flat) faces of a platonic solid & spherical surface to be concentric with that platonic solid. Thus each of identical circles touches other circles exactly at the mid-points of edges of each face of the platonic solid. Then by joining the mid-point of one of the edges (also the point of tangency) to the centre of platonic solid (also the centre of the spherical surface), a right triangle, having its hypotenuse \( R \) and orthogonal sides (i.e. legs) \( r \) & \( h \), is thus obtained. Where,

\[
h = \text{normal distance of each (flat) face from the centre of the platonic solid}
\]

\[
= \text{normal distance of each plane (flat) circle from the centre of the spherical surface}
\]

Here, we are not to derive the parameter such as normal distance of each face from the centre of the different platonic solids. We will prefer to directly use the expressions of normal distance (\( h \)) of each face from the centre of platonic solids from the “Table for the important parameters of all five platonic solids” prepared by the author to derive the mathematical relations of \( R \) & \( r \) in all the cases to be discussed & analysed here. All five cases will be discussed & analysed in the order corresponding to all five platonic solids.

1. Four identical circles, each having a flat radius \( r \), touching one another at 6 different points (i.e. each one touches three other circles) on the whole spherical surface with a radius \( R \) : (Based on Regular Tetrahedron): In this case, let’s assume that each of four identical circles, with a flat radius \( r \), is inscribed by each of four congruent equilateral triangular faces of a regular tetrahedron with an edge length \( a \) such that regular tetrahedron is concentric with the spherical surface having the centre \( O \) & a radius \( R \). In this case, all 6 points of tangency, lying on the spherical surface, are coincident with the mid-points of all 6 edges of a regular tetrahedron. Now, consider one of the four identical circles with the centre \( C \) on the flat face & a flat radius \( r \), touching three other circles at the points \( A \), \( B \) & \( D \) (lying on the spherical surface as well as on the edges of regular tetrahedron) and is inscribed by an equilateral triangular face of regular tetrahedron with an edge length \( a \). (As shown in the figure 1)

We know that the inscribed radius \( r \) & the edge length \( a \) of a regular \( n \)-gon are related as follows

\[
r = \frac{a}{2} \cot \frac{\pi}{n} \Rightarrow a = 2r \tan \frac{\pi}{n}
\]

Figure 1: One of 4 identical circles, with the centre \( C \) on the flat face & a flat radius \( r \), is touching three other circles at the points \( A \), \( B \) & \( D \) (lying on the spherical surface as well as on the edges of face) and is inscribed by an equilateral triangular face of a regular tetrahedron (concentric with the spherical surface) with an edge length \( a \).
Hence, by setting \( n = 3 \) for an equilateral triangular face, we get

\[
a = 2rtan\frac{\pi}{3} \Rightarrow a = 2r\sqrt{3}
\]

Now, we have

\[
h = \text{normal distance of each face from the centre of the regular tetrahedron with edge length } a
\]

\[
= \text{normal distance of plane (flat) circle with centre } C \text{ from the centre } O \text{ of spherical surface}
\]

\[
\Rightarrow h = OC = \frac{a}{2\sqrt{6}} \quad \text{(from the table of platonic solids)}
\]

\[
= \frac{(2r\sqrt{3})}{2\sqrt{6}} = \frac{r}{\sqrt{2}} \quad \text{(by setting the value of edge length } a)\]

\[
\Rightarrow h = OC = \frac{r}{\sqrt{2}}
\]

Draw the perpendicular \( OC \) from the centre \( O \) of the spherical surface (i.e. centre of regular tetrahedron) to the centre \( C \) of the plane (flat) circle & join any of the points \( A, B \) & \( D \) of tangency of the plane circle say point \( A \) (i.e. mid-point of one of the edges of regular tetrahedron) to the centre \( O \) of the spherical surface (i.e. the centre of regular tetrahedron). Thus, we obtain a right \( \Delta OCA \) (as shown in the figure 2)

Applying Pythagoras Theorem in right \( \Delta OCA \) as follows

\[
(OA)^2 = (OC)^2 + (CA)^2 \Rightarrow R^2 = \left(\frac{r}{\sqrt{2}}\right)^2 + r^2 = \frac{3}{2}r^2 \Rightarrow r^2 = \frac{2}{3}R^2
\]

\[
\therefore \text{ flat radius of each circle, } r = R\sqrt{\frac{2}{3}} \approx 0.81649658 R
\]

Arc radius (\( arc \ r \)) of each of 4 identical circles: Consider arc radius \( AC' \) on the spherical surface with a radius \( R \) then we have

In right \( \Delta OCA \)

\[
\sin \theta = \frac{CA}{OA} = \frac{r}{R} = \frac{2}{\sqrt{3}} \Rightarrow \theta = \sin^{-1}\left(\frac{2}{\sqrt{3}}\right)
\]

\[
\Rightarrow \theta = \frac{arc \ AC'}{R} \Rightarrow \text{arc radius } = arc \ AC' = R\theta = R\sin^{-1}\left(\frac{2}{\sqrt{3}}\right)
\]

\[
\therefore \text{ radius of each circle as a great circle arc, } arc \ r = R\sin^{-1}\left(\frac{2}{\sqrt{3}}\right) \approx 0.955316618 R
\]

Total area (\( A_s \)) covered by the four identical circles on the spherical surface: In order to calculate the area covered by each of the four identical circles on the spherical surface with a radius \( R \), let’s first find out the solid angle subtended by each circle with a flat radius \( r \) at the centre \( O \) of the spherical surface (See the figure 2 above) by using formula of solid angle of a right cone concentric with a sphere as follows

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Solid angle subtended by each circle at the centre of the spherical surface, \( \omega = 2\pi(1 - \cos \theta) \)

\[ \Rightarrow \omega = 2\pi \left( 1 - \frac{1}{\sqrt{3}} \right) \quad \text{(setting the value of } \cos \theta = \frac{1}{\sqrt{3}} \text{ from the figure 2 above)} \]

Hence, the total surface area covered by all 4 identical circles on the sphere, is given as

\[ A_s = (\text{no. of circles}) \times (\text{solid angle } \omega) \text{ subtended by each circle}) \times (R^2) = 4 \left( 2\pi \left( 1 - \frac{1}{\sqrt{3}} \right) \right) R^2 \]

\[ A_s = 8\pi R^2 \left( 1 - \frac{1}{\sqrt{3}} \right) \approx 10.62234631 \text{ } R^2 \]

Hence, the percentage of total surface area covered by all 4 identical circles on the sphere, is given as

\[
\% \text{ of total surface area covered} = \frac{\text{total surface area covered by all the circles}}{\text{total surface area of the sphere}} \times 100
\]

\[
= \frac{8\pi R^2 \left( 1 - \frac{1}{\sqrt{3}} \right)}{4\pi R^2} \times 100 = 200 \left( 1 - \frac{1}{\sqrt{3}} \right) \%
\]

\[ \therefore \% \text{ of total surface area covered} = 200 \left( 1 - \frac{1}{\sqrt{3}} \right) \% \approx 84.53 \%
\]

KEY POINT-1: 4 identical circles, touching one another at 6 different points (i.e. each one touches three other circles) on a whole spherical surface, always cover up approximately 84.53% of total surface area irrespective of the radius of the circles or the radius of the spherical surface while approximately 15.47% of total surface area is left uncovered by the circles.

2. Six identical circles, each having a flat radius \( r \), touching one another at 12 different points (i.e. each one touches four other circles) on the whole spherical surface with a radius \( R \) (Based on Regular Hexahedron i.e. Cube) In this case, let’s assume that each of 6 identical circles, with a flat radius \( r \), is inscribed by each of 6 congruent square faces of a cube (regular hexahedron) with an edge length \( a \) such that cube is concentric with the spherical surface having the centre O & a radius R. In this case, all 12 points of tangency, lying on the spherical surface, are coincident with the mid-points of all 12 edges of a regular hexahedron i.e. cube. Now, consider one of 6 identical circles with the centre C on the flat face & a flat radius \( r \), touching four other circles at the points A, B, D & E (lying on the spherical surface as well as on the edges of the cube) and is inscribed by a square face of the cube with an edge length \( a \). (See the figure 3 below)

We know that the inscribed radius \( r \) & the edge length \( a \) of a regular \( n \)-gon are related as follows

\[ r = \frac{a}{2} \cot \frac{\pi}{n} \Rightarrow a = 2rtan \frac{\pi}{n} \]

Hence, by setting \( n = 4 \) for a square face, we get

\[ a = 2rtan \frac{\pi}{4} = a = 2r \]

Now, we have
Draw the perpendicular OC from the centre O of the spherical surface (i.e. centre of the cube) to the centre C of the plane (flat) circle & join any of the points A, B, D & E of tangency of the plane circle say point A (i.e. mid-point of one of the edges of cube) to the centre O of the spherical surface (i.e. the centre of cube). Thus, we obtain a right ΔOCA (as shown in the figure 4 below)

Applying Pythagoras Theorem in right ΔOCA as follows

\[(OA)^2 = (OC)^2 + (CA)^2 \Rightarrow R^2 = (r)^2 + r^2 = 2r^2 \Rightarrow r^2 = \frac{1}{2}R^2\]

\[\therefore \text{flat radius of each circle, } r = \frac{R}{\sqrt{2}} \approx 0.707106781 \]

**Arc radius** (arc \(r\)) of each of 6 identical circles: Consider arc radius \(AC'\) on the spherical surface with a radius \(R\) then we have

In right ΔOCA

\[\sin \theta = \frac{CA}{OA} = \frac{r}{R} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}\]

\[\Rightarrow \theta = \frac{\text{arc } AC'}{R} \Rightarrow \text{arc radius } = \text{arc } AC' = R\theta = \frac{\pi}{4}R\]

\[\therefore \text{radius of each circle as a great circle arc, } \text{arc } r = \frac{\pi R}{4} = 0.785398163 \]

**Total area** \(A_s\) **covered by 6 identical circles on the spherical surface**: In order to calculate the area covered by each of the six identical circles on the spherical surface with a radius \(R\), let's first find out the solid angle subtended by each circle with a flat radius \(r\) at the centre O of the spherical surface (See the figure 4 above) by using **formula of solid angle of a right cone concentric with a sphere** as follows

Solid angle subtended by each circle at the centre of the spherical surface, \(\omega = 2\pi(1 - \cos \theta)\)

\[\Rightarrow \omega = 2\pi \left(1 - \frac{1}{\sqrt{2}}\right) \quad \text{(setting the value of } \cos \theta = \frac{1}{\sqrt{2}} \text{ from the figure 4 above)}\]

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Hence, the total surface area covered by all 6 identical circles on the sphere, is given as

\[ A_s = (\text{no. of circles}) \times (\text{solid angle (\(\omega\)) subtended by each circle}) \times (R^2) = 6 \left(2\pi \left(1 - \frac{1}{\sqrt{2}}\right)\right)R^2 \]

\[ = 12\pi R^2 \left(1 - \frac{1}{\sqrt{2}}\right) \]

\[ \therefore A_s = 12\pi R^2 \left(1 - \frac{1}{\sqrt{2}}\right) \approx 11.04181421 R^2 \]

Hence, the percentage of total surface area covered by all 6 identical circles on the sphere, is given as

\[ \% \text{ of total surface area covered} = \frac{\text{total surface area covered by all the circles}}{\text{total surface area of the sphere}} \times 100 \]

\[ = \frac{12\pi R^2 \left(1 - \frac{1}{\sqrt{2}}\right)}{4\pi R^2} \times 100 = 300 \left(1 - \frac{1}{\sqrt{2}}\right) \% \]

\[ \therefore \% \text{ of total surface area covered} = 300 \left(1 - \frac{1}{\sqrt{2}}\right) \% \approx 87.87 \% \]

**KEY POINT-2:** 6 identical circles, touching one another at 12 different points (i.e. each one touches four other circles) on a whole spherical surface, always cover up approximately 87.87 % of total surface area irrespective of the radius of the circles or the radius of the spherical surface while approximately 12.13 % of total surface area is left uncovered by the circles.

3. Eight identical circles, each having a flat radius \(r\), touching one another at 12 different points (i.e. each one touches three other circles) on the whole spherical surface with a radius \(R\) (Based on Regular Octahedron): In this case, let’s assume that each of 8 identical circles, with a flat radius \(r\), is inscribed by each of 8 congruent equilateral triangular faces of a regular octahedron with an edge length \(a\) such that regular octahedron is concentric with the spherical surface having the centre O & a radius \(R\). In this case, all 12 points of tangency, lying on the spherical surface, are coincident with the mid-points of all 12 edges of a regular octahedron. Now, consider one of 8 identical circles with the centre C on the flat face & a flat radius \(r\), touching three other circles at the points A, B & D (lying on the spherical surface as well as on the edges of regular octahedron) and is inscribed by an equilateral triangular face of regular octahedron with an edge length \(a\) . (As shown in the figure 5)

We know that the inscribed radius \(r\) & the edge length \(a\) of a regular n-gon are related as follows

\[ r = \frac{a}{2} \cot \frac{\pi}{n} \quad \Rightarrow \quad a = 2rtan \frac{\pi}{n} \]

Hence, by setting \(n = 3\) for an equilateral triangular face, we get

\[ a = 2rtan \frac{\pi}{3} \quad \Rightarrow \quad a = 2r\sqrt{3} \]

![Figure 5: One of 8 identical circles, with the centre C on the flat face & a flat radius \(r\), is touching three other circles at the points A, B & D (lying on the spherical surface as well as on the edges of regular octahedron) and is inscribed by an equilateral triangular face of regular octahedron (concentric with the spherical surface) with an edge length \(a\)](image)
Now, we have

\[ h = \text{normal distance of each face from the centre of the regular octahedron with edge length } a \]

\[ = \text{normal distance of plane (flat) circle with centre } C \text{ from the centre } O \text{ of the spherical surface} \]

\[ \Rightarrow h = OC = \frac{a}{\sqrt{6}} \quad \text{(from the table of platonic solids)} \]

\[ = \frac{(2r\sqrt{3})}{\sqrt{6}} = r\sqrt{2} \quad \text{(by setting the value of edge length } a) \]

\[ \Rightarrow h = OC = r\sqrt{2} \]

Draw the perpendicular OC from the centre O of the spherical surface (i.e. centre of regular octahedron) to the centre C of the plane (flat) circle & join any of the points A, B & D of tangency of the plane circle say point A (i.e. mid-point of one of the edges of regular octahedron) to the centre O of the spherical surface (i.e. the centre of regular octahedron). Thus, we obtain a right ΔOCA (as shown in the figure 6)

Applying Pythagoras Theorem in right ΔOCA as follows

\[ (OA)^2 = (OC)^2 + (CA)^2 \quad \Rightarrow R^2 = (r\sqrt{2})^2 + r^2 = 3r^2 \quad \Rightarrow r^2 = \frac{1}{3}R^2 \]

\[ \therefore \text{ flat radius of each circle, } r = \frac{R}{\sqrt{3}} \approx 0.577350269\ R \]

**Arc radius (arc } r) \text{ of each of } 8 \text{ identical circles:} \text{ Consider arc radius } AC' \text{ on the spherical surface with a radius } R \text{ then we have} \]

In right ΔOCA

\[ \sin\theta = \frac{CA}{OA} = \frac{r}{R} = \frac{1}{\sqrt{3}} \quad \Rightarrow \theta = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right) \]

\[ \Rightarrow \theta = \frac{\text{arc } AC'}{R} \quad \Rightarrow \text{arc radius } = \text{arc } AC' = R\theta = R\sin^{-1}\left(\frac{1}{\sqrt{3}}\right) \]

\[ \therefore \text{ radius of each circle as a great circle arc, } \text{ arc } r = R\sin^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 0.615479708\ R \]

**Total area \((A_{s})\) covered by } 8 \text{ identical circles on the spherical surface:} \text{ In order to calculate the area covered by each of the eight identical circles on the spherical surface with a radius } R, \text{ let's first find out the solid angle subtended by each circle with a flat radius } r \text{ at the centre } O \text{ of the spherical surface (See the figure 6 above) by using formula of solid angle of a right cone concentric with a sphere} \text{ as follows} \]

Solid angle subtended by each circle at the centre of the spherical surface, \(\omega = 2\pi(1 - \cos\theta)\)

\[ \Rightarrow \omega = 2\pi \left(1 - \frac{2}{\sqrt{3}}\right) \quad \left(\text{setting the value of } \cos\theta = \sqrt{\frac{2}{3}} \text{ from the figure 6 above} \right) \]

Hence, the total surface area covered by } 8 \text{ identical circles on the sphere, is given as
\[ A_s = (\text{no. of circles}) \times (\text{solid angle } \omega \text{ subtended by each circle}) \times (R^2) = 8 \left(2\pi \left(1 - \frac{2}{\sqrt{3}}\right)\right) R^2 \]

\[ = 16\pi R^2 \left(1 - \frac{2}{\sqrt{3}}\right) \]

\[ \therefore A_s = 16\pi R^2 \left(1 - \frac{2}{\sqrt{3}}\right) \approx 9.223887892 \, R^2 \]

Hence, the percentage of total surface area covered by all 8 identical circles on the sphere, is given as

\[ \% \text{ of total surface area covered} = \frac{\text{total surface area covered by all the circles}}{\text{total surface area of the sphere}} \times 100 \]

\[ = \frac{16\pi R^2 \left(1 - \frac{2}{\sqrt{3}}\right)}{4\pi R^2} \times 100 = 400 \left(1 - \frac{2}{\sqrt{3}}\right) \% \]

\[ \therefore \% \text{ of total surface area covered} = 400 \left(1 - \frac{2}{\sqrt{3}}\right) \% \approx 73.4 \% \]

KEY POINT-3: 8 identical circles, touching one another at 12 different points (i.e. each one touches three other circles) on a whole spherical surface, always cover up approximately 73.4 % of total surface area irrespective of the radius of the circles or the radius of the spherical surface while approximately 26.6 % of total surface area is left uncovered by the circles.

4. Twelve identical circles, each having a flat radius \( r \), touching one another at 30 different points (i.e. each one touches five other circles) on the whole spherical surface with a radius \( R \) (Based on Regular dodecahedron): In this case, let’s assume that each of 12 identical circles, with a flat radius \( r \), is inscribed by each of 12 congruent regular pentagonal faces of a regular dodecahedron with an edge length \( a \) such that regular dodecahedron is concentric with the spherical surface having the centre \( O \) & a radius \( R \). In this case, all 30 points of tangency, lying on the spherical surface, are coincident with the mid-points of all 30 edges of a regular dodecahedron. Now, consider one of the 12 identical circles with the centre \( C \) on the flat face & a flat radius \( r \), touching five other circles at the points \( A, B, D, E \) & \( F \) (lying on the spherical surface as well as on the edges of the dodecahedron) and is inscribed by a regular pentagonal face of the dodecahedron with an edge length \( a \). (See the figure 7 below)

We know that the inscribed radius \( r \) & the edge length \( a \) of a regular \( n \)-gon are related as follows

\[ r = \frac{a}{2} \cot \frac{\pi}{n} \Rightarrow a = 2r \tan \frac{\pi}{n} \]

Hence, by setting \( n = 4 \) for a square face, we get

\[ a = 2r \tan \frac{\pi}{5} \Rightarrow a = 2r \sqrt{5 - 2\sqrt{5}} \]

Now, we have

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Draw the perpendicular $OC$ from the centre $O$ of the spherical surface (i.e. centre of the regular dodecahedron) to the centre $C$ of the plane (flat) circle & join any of the points $A$, $B$, $D$, $E$ & $F$ of tangency of the plane circle say point $A$ (i.e. mid-point of one of the edges of dodecahedron) to the centre $O$ of the spherical surface (i.e. the centre of dodecahedron). Thus, we obtain a right $\Delta OCA$ (as shown in the figure 8 below)

Applying Pythagoras Theorem in right $\Delta OCA$ as follows

\[
(OA)^2 = (OC)^2 + (CA)^2 \quad \Rightarrow R^2 = \left(\frac{(1 + \sqrt{5})r}{2}\right)^2 + r^2 = r^2 \left(\frac{6 + 2\sqrt{5} + 4}{4}\right) = r^2 \left(\frac{5 + \sqrt{5}}{2}\right)
\]

\[
\Rightarrow r^2 = R^2 \left(\frac{2}{5 + \sqrt{5}}\right) = R^2 \left(\frac{2(5 - \sqrt{5})}{(5 + \sqrt{5})(5 - \sqrt{5})}\right) = R^2 \left(\frac{2(5 - \sqrt{5})}{20}\right) = R^2 \left(\frac{5 - \sqrt{5}}{10}\right)
\]

\[
\therefore \text{ flat radius of each circle, } \ r = R \left(\frac{5 - \sqrt{5}}{10}\right) = 0.525731112 R
\]

Arc radius ($arc \ r$) of each of 12 identical circles: Consider arc radius $AC'$ on the spherical surface with a radius $R$ then we have

In right $\Delta OCA$

\[
\sin \theta = \frac{CA}{OA} = \frac{r}{R} = \frac{5 - \sqrt{5}}{10} \quad \Rightarrow \theta = \sin^{-1}\left(\frac{5 - \sqrt{5}}{10}\right)
\]
\[ \Rightarrow \theta = \frac{arc\ AC'}{R} \Rightarrow arc\ radius\ =\ arc\ AC' = R\theta = R\sin^{-1}\left(\frac{5 - \sqrt{5}}{10}\right) \]

\[ \therefore \text{radius of each circle as a great circle arc, } \ arcs = R\sin^{-1}\left(\frac{5 - \sqrt{5}}{10}\right) \approx 0.553574358\ R \]

Total area \(A_s\) covered by 12 identical circles on the spherical surface: In order to calculate the area covered by each of 12 identical circles on the spherical surface with a radius \(R\), let’s first find out the solid angle subtended by each circle with a flat radius \(r\) at the centre \(O\) of the spherical surface (see the figure 8) by using formula of solid angle of a right cone concentric with a sphere as follows

\[ \omega = 2\pi(1 - \cos\theta) \]

\[ \Rightarrow \omega = 2\pi\left(1 - \frac{5 + \sqrt{5}}{\sqrt{10}}\right) \quad \text{setting the value of } \cos\theta = \frac{5 + \sqrt{5}}{\sqrt{10}} \text{ from fig.8} \]

Hence, the total surface area covered by all 12 identical circles on the sphere, is given as

\[ A_s = (\text{no. of circles}) \times (\text{solid angle } \omega \text{ subtended by each circle}) \times (R^2) \]

\[ = 12\left(2\pi\left(1 - \frac{5 + \sqrt{5}}{\sqrt{10}}\right)\right)R^2 = 24\pi R^2\left(1 - \frac{5 + \sqrt{5}}{\sqrt{10}}\right) \]

\[ \therefore A_s = 24\pi R^2\left(1 - \frac{5 + \sqrt{5}}{\sqrt{10}}\right) \approx 11.26066376\ R^2 \]

Hence, the percentage of total surface area covered by all 12 identical circles on the sphere, is given as

\[ \% \text{ of total surface area covered} = \frac{\text{total surface area covered by all the circles}}{\text{total surface area of the sphere}} \times 100 \]

\[ = \frac{24\pi R^2\left(1 - \frac{5 + \sqrt{5}}{\sqrt{10}}\right)}{4\pi R^2} \times 100 = 600\left(1 - \frac{5 + \sqrt{5}}{\sqrt{10}}\right)\% \]

\[ \therefore \% \text{ of total surface area covered} = 600\left(1 - \frac{5 + \sqrt{5}}{\sqrt{10}}\right)\% \approx 89.61\% \]

KEY POINT-4: 12 identical circles, touching one another at 30 different points (i.e. each one touches five other circles) on a whole spherical surface, always cover up approximately 89.61% of total surface area irrespective of the radius of the circles or the radius of the spherical surface while approximately 10.39% of total surface area is left uncovered by the circles.

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5. Twenty identical circles, each having a flat radius \( r \), touching one another at 30 different points (i.e. each one touches three other circles) on the whole spherical surface with a radius \( R \) (Based on Regular Icosahedron): In this case, let’s assume that each of 20 identical circles, with a flat radius \( r \), is inscribed by each of 20 congruent equilateral triangular faces of a regular icosahedron with an edge length \( a \) such that regular icosahedron is concentric with the spherical surface having the centre O & a radius \( R \). In this case, all 30 points of tangency, lying on the spherical surface, are coincident with the mid-points of all 30 edges of a regular icosahedron. Now, consider one of the 12 identical circles with the centre C on the flat face & a flat radius \( r \), touching three other circles at the points A, B & D (lying on the spherical surface as well as on the edges of regular icosahedron) and is inscribed by an equilateral triangular face of regular icosahedron with an edge length \( a \). (As shown in the figure 9)

We know that the inscribed radius \( r \) & the edge length \( a \) of a regular \( n \)-gon are related as follows

\[
r = \frac{a}{2} \cot \frac{\pi}{n} \Rightarrow a = 2rtan\frac{\pi}{n}
\]

Hence, by setting \( n = 3 \) for an equilateral triangular face, we get

\[
a = 2rtan\frac{\pi}{3} \Rightarrow a = 2r\sqrt{3}
\]

Now, we have

\[h = \text{normal distance of each face from the centre of the regular icosahedron with edge length } a\]
\[= \text{normal distance of plane (flat) circle with centre C from the centre O of the spherical surface}\]

\[\Rightarrow h = OC = \frac{a(3 + \sqrt{5})}{4\sqrt{3}} \quad (\text{from the table of platonic solids})\]
\[= \frac{(3 + \sqrt{5})(2r\sqrt{3})}{4\sqrt{3}} = \frac{(3 + \sqrt{5})r}{2} \quad (\text{by setting the value of edge length } a)\]

\[\Rightarrow h = OC = \frac{(3 + \sqrt{5})r}{2}\]

Draw the perpendicular OC from the centre O of the spherical surface (i.e. centre of regular icosahedron) to the centre C of the plane (flat) circle & join any of the points A, B & D of tangency of the plane circle say point A (i.e. mid-point of one of the edges of regular icosahedron) to the centre O of the spherical surface (i.e. the centre of regular icosahedron). Thus, we obtain a right \( \Delta OCA \) (as shown in the figure 10 below)

Applying Pythagoras Theorem in right \( \Delta OCA \) as follows

\[(OA)^2 = (OC)^2 + (CA)^2 \quad \Rightarrow R^2 = \left(\frac{(3 + \sqrt{5})r}{2}\right)^2 + r^2 = r^2 \left(\frac{14 + 2\sqrt{5} + 4}{4}\right) = r^2 \left(\frac{9 + \sqrt{5}}{2}\right)\]
\[ r^2 = R^2 \left( \frac{2}{9 + \sqrt{5}} \right) = R^2 \left( \frac{2(9 - \sqrt{5})}{(9 + \sqrt{5})(9 - \sqrt{5})} \right) = R^2 \left( \frac{2(9 - \sqrt{5})}{81 - 5} \right) \]
\[ = R^2 \left( \frac{9 - \sqrt{5}}{38} \right) \Rightarrow r = R \sqrt{\frac{9 - \sqrt{5}}{38}} \]

**Flat radius of each circle,** \( r = R \sqrt{\frac{9 - \sqrt{5}}{38}} \approx 0.421898342 R \)

**Arc radius \((arc r)\) of each of 20 identical circles:** Consider arc radius \(AC'\) on the spherical surface with a radius \(R\) then we have

In right \(\triangle OCA\)

\[ \sin\theta = \frac{CA}{OA} = \frac{r}{R} = \frac{\sqrt{9 - \sqrt{5}}}{38} \Rightarrow \theta = \sin^{-1} \left( \frac{\sqrt{9 - \sqrt{5}}}{38} \right) \]

\[ \Rightarrow \theta = \frac{arc \ AC'}{R} \Rightarrow arc \ radius \ AC' = R\theta = R \sin^{-1} \left( \frac{\sqrt{9 - \sqrt{5}}}{38} \right) \]

**Radius of each circle as a great circle arc,** \(arc r = R \sin^{-1} \left( \frac{\sqrt{9 - \sqrt{5}}}{38} \right) \approx 0.435538116 R\)

**Total area \((A_s)\) covered by 20 identical circles on the spherical surface:** In order to calculate the area covered by each of 20 identical circles on the spherical surface with a radius \(R\), let’s first find out the solid angle subtended by each circle with a flat radius \(r\) at the centre O of the spherical surface (See the figure 10 above) by using formula of solid angle of a right cone concentric with a sphere as follows

Solid angle subtended by each circle at the centre of the spherical surface, \(\omega = 2\pi(1 - \cos\theta)\)

\[ \Rightarrow \omega = 2\pi \left( 1 - \frac{29 + \sqrt{5}}{38} \right) \quad (setting \ the \ value \ of \ \cos\theta = \frac{29 + \sqrt{5}}{38} \ from \ the \ figure \ 10 \ above) \]

Hence, the **total surface area covered by all 20 identical circles on the sphere**, is given as

\[ A_s = (no. \ of \ circles) \times (solid \ angle \ (\omega) \ subtended \ by \ each \ circle) \times (R^2) \]

\[ = 20 \left( 2\pi \left( 1 - \frac{29 + \sqrt{5}}{38} \right) \right) R^2 = 40\pi R^2 \left( 1 - \frac{29 + \sqrt{5}}{38} \right) \]

\[ \therefore A_s = 40\pi R^2 \left( 1 - \frac{29 + \sqrt{5}}{38} \right) \approx 11.73156864 \ R^2 \]
Hence, the percentage of total surface area covered by all 20 identical circles on the sphere, is given as

\[
\text{% of total surface area covered} = \frac{\text{total surface area covered by all the circles}}{\text{total surface area of the sphere}} \times 100
\]

\[
= \frac{40\pi R^2 \left(1 - \frac{29 + \sqrt{5}}{38}\right)}{4\pi R^2} \times 100 = 1000 \left(1 - \frac{29 + \sqrt{5}}{38}\right) \%
\]

\[\therefore \text{% of total surface area covered} = 1000 \left(1 - \frac{29 + \sqrt{5}}{38}\right) \% \approx 93.36 \%
\]

**KEY POINT-5:** 20 identical circles, touching one another at 30 different points (i.e. each one touches three other circles) on a whole spherical surface, always cover up approximately 93.36 % of the total surface area irrespective of the radius of the circles or the radius of the spherical surface while approximately 6.64 % of total surface area is left uncovered by the circles.

Thus, 8 identical circles, touching one another at 12 different points (each one touches three other circles), cover up the minimum approximately 73.4 % while 20 identical circles, touching one another at 30 different points (each one touches three other circles), cover up the maximum approximately 93.36 % of the total surface area of a sphere.

**Conclusion:** Let there a certain no. of the identical circles touching one another on a whole spherical surface with a radius \(R\) then we can easily find out the important parameters such as flat radius & arc radius of each circle and surface area covered by the circles on the sphere as tabulated below

<table>
<thead>
<tr>
<th>Total no. of identical circles touching one another on a whole spherical surface with a radius (R)</th>
<th>No. of circles touching each circle</th>
<th>Total no. of points of tangency</th>
<th>Flat radius of each circle (i.e. radius of each plane (flat) circle)</th>
<th>Arc radius of each circle (i.e. radius of each circle as a great circle arc on spherical surface)</th>
<th>Total surface area covered by all the circles</th>
<th>Percentage (%) of total surface area covered by all the circles</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>3</td>
<td>6</td>
<td>(\sqrt{\frac{2}{3}} R) \approx 0.81649658 (R)</td>
<td>(R \sin^{-1}\left(\sqrt{\frac{2}{3}}\right)) \approx 0.955316618 (R)</td>
<td>(8\pi R^2 \left(1 - \frac{1}{\sqrt{3}}\right)) \approx 10.62234631 (R^2)</td>
<td>84.53 %</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>12</td>
<td>(\sqrt{\frac{2}{3}} R) \approx 0.707106781 (R)</td>
<td>(\frac{\pi R}{4}) \approx 0.785398163 (R)</td>
<td>(12\pi R^2 \left(1 - \frac{1}{\sqrt{2}}\right)) \approx 11.04181421 (R^2)</td>
<td>87.87 %</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>12</td>
<td>(\sqrt{\frac{2}{3}} R) \approx 0.577350269 (R)</td>
<td>(R \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)) \approx 0.615479708 (R)</td>
<td>(16\pi R^2 \left(1 - \frac{2}{\sqrt{3}}\right)) \approx 9.223887892 (R^2)</td>
<td>73.4 %</td>
</tr>
<tr>
<td>12</td>
<td>5</td>
<td>30</td>
<td>(\sqrt{\frac{5 - \sqrt{5}}{10}} R) \approx 0.525731112 (R)</td>
<td>(R \sin^{-1}\left(\frac{5 - \sqrt{5}}{10}\right)) \approx 0.553574358 (R)</td>
<td>(24\pi R^2 \left(1 - \frac{5 + \sqrt{5}}{10}\right)) \approx 11.26066376 (R^2)</td>
<td>89.61 %</td>
</tr>
</tbody>
</table>

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\[ R \left( \frac{9 - \sqrt{5}}{38} \right) \approx 0.421898342 R \]

\[ R \sin^{-1} \left( \frac{9 - \sqrt{5}}{38} \right) \approx 0.435538116 R \]

\[ 40\pi R^2 \left( 1 - \frac{29 + \sqrt{5}}{38} \right) \approx 11.73156864 R^2 \]

Note: Above articles had been derived & illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)

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