Derivations of inscribed & circumscribed radii for three externally touching circles

Mathematical Analysis of Three Externally Touching Circles

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1. Introduction:

Consider three circles having centres A, B & C and radii \( a, b \) & \( c \) respectively, touching each other externally such that a small circle \( P \) is inscribed in the gap & touches them externally & a large circle \( Q \) circumscribes them & is touched by them internally. We are to calculate the radii of inscribed circle \( P \) (touching three circles with centres A, B & C externally) & circumscribed circle \( Q \) (touched by three circles with centres A, B & C internally) (See figure 1)

2. Derivation of the radius of inscribed circle:

Let \( r \) be the radius of inscribed circle, with centre \( O \), externally touching the given circles, having centres A, B & C and radii \( a, b \) & \( c \), at the points M, N & P respectively. Now join the centre \( O \) to the centres A, B & C by dotted straight lines to obtain \( \Delta AOB, \Delta BOC \) & \( \Delta AOC \) & also join the centres A, B & C by dotted straight lines to obtain \( \Delta ABC \) (As shown in the figure 2 below) Thus we have

\[
AM = a, \quad BN = b, \quad CP = c \quad & \\
OM = ON = OP = r \quad (radius \ of \ inscribed \ circle)
\]

\[
\text{In } \Delta ABC
\]

\[
AB = a + b, \quad BC = b + c \quad & \quad AC = a + c
\]

\[\Rightarrow \text{semiperimeter} = \frac{AB + BC + AC}{2} = \frac{a + b + b + c + a + c}{2} = a + b + c
\]

\[
\sin \frac{\angle ACB}{2} = \sqrt{\frac{(s - AC)(s - BC)}{AC(BC)}}
\]

\[\Rightarrow \sin \frac{\alpha}{2} = \sqrt{\frac{(a + b + c - a - c)(a + b + c - b - c)}{(a + c)(b + c)}}
\]

\[
\sin \frac{\alpha}{2} = \sqrt{\frac{ab}{(a + c)(b + c)}} \quad \ldots \ldots \text{(I)}
\]

Similarly, in \( \Delta BOC \)

\[\text{semiperimeter} = \frac{OB + BC + OC}{2} \Rightarrow s = \frac{r + b + b + c + r + c}{2} = b + c + r
\]
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\[
\cos \frac{\angle BCO}{2} = \frac{s(s - OB)}{(BC)(OC)} \Rightarrow \cos \frac{\alpha_1}{2} = \sqrt{\frac{(b + c + r)(b + c + r - b)}{(b + c)(r + c)}} = \frac{c(b + c + r)}{(b + c)(c + r)}
\]

\[
\Rightarrow \cos \frac{\alpha_1}{2} = \sqrt{\frac{c(b + c + r)}{(b + c)(c + r)}} \quad \ldots \ldots \ldots \ldots (II)
\]

Similarly, in \(\triangle AOC\)

\[
\text{semiperimeter} = \frac{OA + AC + OC}{2} \Rightarrow s = \frac{r + a + a + c + r + c}{2} = a + c + r
\]

\[
\cos \frac{\angle ACO}{2} = \frac{s(s - OA)}{(AC)(OC)} \Rightarrow \cos \frac{\alpha_2}{2} = \sqrt{\frac{(a + c + r)(a + c + r - a)}{(a + c)(r + c)}} = \frac{c(a + c + r)}{(a + c)(c + r)}
\]

\[
\Rightarrow \cos \frac{\alpha_2}{2} = \sqrt{\frac{c(a + c + r)}{(a + c)(c + r)}} \quad \ldots \ldots \ldots \ldots (III)
\]

Now, again in \(\triangle ABC\), we have

\[
\angle ACO + \angle BCO = \angle ACB \Rightarrow \alpha_2 + \alpha_1 = \alpha \text{ or } \frac{\alpha_1 + \alpha_2}{2} = \frac{\alpha}{2}
\]

Now, taking \cosines of both the sides we have

\[
\cos \left(\frac{\alpha_1 + \alpha_2}{2}\right) = \cos \frac{\alpha}{2} \Rightarrow \cos \frac{\alpha_1}{2}\cos \frac{\alpha_2}{2} - \sin \frac{\alpha_1}{2}\sin \frac{\alpha_2}{2} = \cos \frac{\alpha}{2}
\]

\[
\Rightarrow \cos^2 \frac{\alpha_1}{2}\cos^2 \frac{\alpha_2}{2} + \sin^2 \frac{\alpha_1}{2}\sin^2 \frac{\alpha_2}{2} - 2\sin \frac{\alpha_1}{2}\sin \frac{\alpha_2}{2}\cos \frac{\alpha_1}{2}\cos \frac{\alpha_2}{2} = \cos^2 \frac{\alpha}{2}
\]

\[
\Rightarrow \cos^2 \frac{\alpha_1}{2}\cos^2 \frac{\alpha_2}{2} + \left(1 - \cos^2 \frac{\alpha_1}{2}\right) \left(1 - \cos^2 \frac{\alpha_2}{2}\right) - \cos^2 \frac{\alpha}{2} = 2\sin \frac{\alpha_1}{2}\sin \frac{\alpha_2}{2}\cos \frac{\alpha_1}{2}\cos \frac{\alpha_2}{2}
\]

\[
\Rightarrow 2\cos^2 \frac{\alpha_1}{2}\cos^2 \frac{\alpha_2}{2} - \cos^2 \frac{\alpha_1}{2} - \cos^2 \frac{\alpha_2}{2} + \sin^2 \frac{\alpha}{2} = 2\cos \frac{\alpha_1}{2}\cos \frac{\alpha_2}{2}\sqrt{\left(1 - \cos^2 \frac{\alpha_1}{2}\right)\left(1 - \cos^2 \frac{\alpha_2}{2}\right)}
\]

\[
\Rightarrow \left(2\cos^2 \frac{\alpha_1}{2}\cos^2 \frac{\alpha_2}{2} - \cos^2 \frac{\alpha_1}{2} - \cos^2 \frac{\alpha_2}{2} + \sin^2 \frac{\alpha}{2}\right)^2 = 4\cos^2 \frac{\alpha_1}{2}\cos^2 \frac{\alpha_2}{2}\left(1 - \cos^2 \frac{\alpha_1}{2}\right)\left(1 - \cos^2 \frac{\alpha_2}{2}\right)
\]

\[
\Rightarrow 4\cos^4 \frac{\alpha_1}{2}\cos^4 \frac{\alpha_2}{2} - 2\cos^4 \frac{\alpha_1}{2}\cos^2 \frac{\alpha_2}{2} - 2\cos^4 \frac{\alpha_1}{2}\cos^2 \frac{\alpha_2}{2} + 2\cos^2 \frac{\alpha_1}{2}\cos^4 \frac{\alpha_2}{2} + 2\cos^2 \frac{\alpha_1}{2}\cos^4 \frac{\alpha_2}{2} + 2\cos^4 \frac{\alpha_1}{2}\cos^2 \frac{\alpha_2}{2}
\]

\[
\Rightarrow \cos^4 \frac{\alpha_1}{2} + \cos^4 \frac{\alpha_2}{2} - \cos^2 \frac{\alpha_1}{2}\sin^2 \frac{\alpha}{2} - 2\cos^2 \frac{\alpha_1}{2}\cos^2 \frac{\alpha_2}{2} + \cos \frac{\alpha_1}{2}\cos \frac{\alpha_2}{2}
\]

\[
\Rightarrow \cos^4 \frac{\alpha_2}{2} - \cos^2 \frac{\alpha_1}{2}\sin^2 \frac{\alpha}{2} + 2\cos^2 \frac{\alpha_1}{2}\cos^2 \frac{\alpha_2}{2} - \cos^2 \frac{\alpha_1}{2}\sin^2 \frac{\alpha}{2} - \cos^2 \frac{\alpha_2}{2}\sin^2 \frac{\alpha}{2}
\]

\[
\Rightarrow \sin^4 \frac{\alpha}{2} = 4\cos^4 \frac{\alpha_1}{2}\cos^4 \frac{\alpha_2}{2} - 4\cos^4 \frac{\alpha_1}{2}\cos^2 \frac{\alpha_2}{2} - 4\cos^2 \frac{\alpha_1}{2}\cos^4 \frac{\alpha_2}{2} + 4\cos^2 \frac{\alpha_1}{2}\cos^2 \frac{\alpha_2}{2}
\]
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Now, substituting all the corresponding values from eq(I), (II) & (III) in above expression, we have

\[
\left(\frac{c(b+c+r)}{(b+c)(c+r)}\right)^4 + \left(\frac{c(a+c+r)}{(a+c)(a+r)}\right)^4 + \left(\frac{ab}{(a+c)(b+c)}\right)^4
\]
\[
+ \left(\frac{c(b+c+r)}{(b+c)(c+r)}\right)^2 \left(\frac{c(a+c+r)}{(a+c)(c+r)}\right)^2 \left(\frac{4ab}{(a+c)(b+c)}\right)^2 - 2
\]
\[
- 2 \left(\frac{ab}{(a+c)(b+c)}\right)^2 \left(\frac{c(b+c+r)}{(b+c)(c+r)}\right)^2 \left(\frac{c(a+c+r)}{(a+c)(c+r)}\right)^2 = 0
\]

Now, on multiplying the above equation by \((a+c)^2(b+c)^2(c+r)^2\), we get
\[
c^2(a+c)^2(b+c)^2(c+r)^2 + c^2(a+c+r)^2(b+c)^2(a+c+r)^2 + a^2b^2(c+r)^2
\]
\[
+ c^2(a+c+r)(b+c+r)(2ab - 2bc - 2ac - 2c^2)
\]
\[
- 2abc(c+r)((a+c)(b+c)+ (b+c)(a+c+r)) = 0
\]
\[
\Rightarrow c^2(a+c)^2(r^2 + 2(b+c)r + (b+c)^2) + c^2(b+c)^2(r^2 + 2(a+c)r + (a+c)^2) + a^2b^2(r^2 + 2cr + c^2)
\]
\[
+ c^2(2ab - 2bc - 2ac - 2c^2)(r^2 + (a+b+c)r + (a+c)(b+c))
\]
\[
- 2abc(c+r)((a+b+c)(b+c)) = 0
\]
\[
\Rightarrow (c^2(a+c)^2 + a^2b^2 + 2c^2(ab - bc - ac - c^2)r^2
\]
\[
+ (2c^2(a+c)(b+c)(a+b+2c) + 2abc^2 + 2c^2(a+b+2c)(ab - bc - ac - c^2))r
\]
\[
+ 2c^2(a+c)^2(b+c)^2 + a^2b^2c^2 + 2c^2(a+c)(b+c)(ab - bc - ac - c^2)
\]
\[
- 2abc(a+b+2c)r^2 - 2abc(2a+c)(b+c) + c(a+b+2c)r - 4abc^2(a+c)(b+c)
\]
\[
= 0
\]
\[
\Rightarrow (a^2c^2 + a^4 + 2ac^3 + b^2c^2 + c^4 + 2bc^3 + a^2b^2 + 2abc^2 - 2bc^3 - 2ac^3 - 2c^4 - 2a^2bc - 2abc^2
\]
\[
+ (2a^2bc + 2a^2c^3 + 2bc^4 + 2c^5 + 4abc^3 + 4ac^4 + 2ab^2c^2 + 2b^2c^3 + 2a^2c + 2c^5
\]
\[
+ 4abc + 4bc^4 + 2a^2b^2c - 2abc^3 - 2ac^3 - 2a^2^3 - 2abc^2 - 2b^2c^3
\]
\[
- 2abc - 2bc^4 + 4abc^3 - 4ac^4 - 4c^5 - 4a^2b^2c - 4a^2bc^2 - 4a^2bc^2 - 8abc^3)r
\]
\[
+ (a^2b^2c^2 + 4a^2b^2c^2 + 4abc^2 + 4a^2bc^3 + 4ac^4 + 4a^2b^2c^2 - 4a^2b^2c^2 - 4a^2b^2c^2
\]
\[
- 4abc^4) = 0
\]
\[
\Rightarrow (a^2b^2 + b^2c^2 + c^2a^2 - 2abc(a+b+c)r^2 - 2abc(ab+bc+ca)r + a^2b^2c^2 = 0
\]

Now, solving the above quadratic equation for the values of \(r\) as follows
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\[
r = \frac{2abc(ab + bc + ca) \pm \sqrt{(2abc(ab + bc + ca))^2 - 4(a^2b^2c^2)(a^2b^2 + b^2c^2 + c^2a^2 - 2abc(a + b + c))}}{2(a^2b^2 + b^2c^2 + c^2a^2 - 2abc(a + b + c))}
\]

\[
= \frac{2abc(ab + bc + ca) \pm 2abc\sqrt{a^2b^2c^2 + 4ab^2c + 4abc^2}}{2(a^2b^2 + b^2c^2 + c^2a^2 - 2abc(a + b + c))} = \frac{abc(ab + bc + ca) \pm abc\sqrt{4abc(a + b + c)}}{(ab + bc + ca)^2 - 4abc(a + b + c)}
\]

\[
\Rightarrow r = abc \left( \frac{(ab + bc + ca) \pm 2\sqrt{abc(a + b + c)}}{(ab + bc + ca)^2 - (2\sqrt{abc(a + b + c)})^2} \right)
\]

Case 1: Taking positive sign, we get

\[
r = abc \left( \frac{(ab + bc + ca) + 2\sqrt{abc(a + b + c)}}{(ab + bc + ca)^2 - (2\sqrt{abc(a + b + c)})^2} \right) = abc \left( \frac{1}{(ab + bc + ca) - 2\sqrt{abc(a + b + c)}} \right)
\]

\[
\Rightarrow r < 0 \ \forall \ a, b, c > 0 \ \text{but} \ r > 0 \ \text{hence this value of radius} \ r \ \text{is discarded}
\]

Case 2: Taking negative sign, we get

\[
r = abc \left( \frac{(ab + bc + ca) - 2\sqrt{abc(a + b + c)}}{(ab + bc + ca)^2 - (2\sqrt{abc(a + b + c)})^2} \right) = abc \left( \frac{1}{(ab + bc + ca) + 2\sqrt{abc(a + b + c)}} \right)
\]

\[
\Rightarrow r > 0 \ \forall \ a, b, c > 0 \ \text{hence this value of radius} \ r \ \text{is accepted}
\]

Hence, the radius \((r)\) of inscribed circle is given as

\[
r = \frac{abc}{2\sqrt{abc(a + b + c) + (ab + bc + ca)}} \ \ (r > 0 \ \forall \ a, b, c > 0)
\]

Above is the required expression to compute the radius \((r)\) of the inscribed circle which externally touches three given circles with radii \(a, b \ & c\) touching each other externally.

3. Derivation of the radius of circumscribed circle: Let \(R\) be the radius of circumscribed circle, with centre \(O\), is internally touched by the given circles, having centres \(A, B \ & C\) and radii \(a, b \ & c\), at the points \(M, N \ & P\) respectively. Now join the centre \(O\) to the centres \(A, B \ & C\) by dotted straight lines to obtain \(\Delta AOB, \Delta BOC \ & \Delta AOC\) & also join the centres \(A, B \ & C\) by dotted straight lines to obtain \(\Delta ABC\) (As shown in the figure 3) Thus we have

\[
AM = a, \quad BN = b, \quad CP = c \ & \quad OM = ON = OP = R \quad (\text{radius of circumscribed circle})
\]

\[
OA = OM - AM = R - a, \quad OB = R - b \ & \quad OC = R - c
\]

In \(\Delta AOB\)

\[
OA = R - a, \quad AB = a + b \ & \quad OB = R - b
\]

\[
\text{semiperimeter} = \frac{OA + AB + OB}{2}
\]

Figure 3: The centres A, B, C & O are joined to each other by dotted straight lines to obtain \(\Delta ABC, \Delta AOB, \Delta BOC \ & \Delta AOC\)

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\[ s = \frac{R - a + a + b + R - b}{2} = R \]

\[ \sin \frac{\angle AOB}{2} = \sqrt{\frac{(s - OA)(s - OB)}{(OA)(OB)}} = \sqrt{\frac{(R - R + a)(R - R + b)}{(R - a)(R - b)}} = \sqrt{\frac{ab}{(R - a)(R - b)}} \quad \text{let} \ \angle AOB = \alpha \]

\[ \Rightarrow \sin \frac{\alpha}{2} = \frac{\sqrt{ab}}{\sqrt{(R - a)(R - b)}} \quad \ldots \ldots \ldots (I) \]

Similarly, in \( \triangle BOC \)

\[ \text{semiperimeter} = \frac{OB + BC + OC}{2} \]

\[ s = \frac{R - b + b + c + R - c}{2} = R \]

\[ \cos \frac{\angle BOC}{2} = \sqrt{\frac{s(s - BC)}{(OB)(OC)}} \Rightarrow \cos \frac{\alpha_1}{2} = \sqrt{\frac{R(R - b - c)}{(R - b)(R - c)}} \quad \ldots \ldots \ldots (II) \]

Similarly, in \( \triangle AOC \)

\[ \text{semiperimeter} = \frac{OA + AC + OC}{2} \Rightarrow s = \frac{R - a + a + c + R - c}{2} = R \]

\[ \cos \frac{\angle AOC}{2} = \sqrt{\frac{s(s - AC)}{(OA)(OC)}} \Rightarrow \cos \frac{\alpha_2}{2} = \sqrt{\frac{R(R - a - c)}{(R - a)(R - c)}} \quad \ldots \ldots \ldots (III) \]

Now, again in \( \triangle AOB \), we have

\[ \angle BOC + \angle AOC = \angle AOB \Rightarrow \alpha_2 + \alpha_1 = \alpha \quad \text{or} \quad \alpha_1 + \alpha_2 = \frac{\alpha}{2} \]

Now, taking \text{cosines} of both the sides we have

\[ \cos \left( \frac{\alpha_1}{2} + \frac{\alpha_2}{2} \right) = \cos \frac{\alpha}{2} \Rightarrow \cos \frac{\alpha_1}{2} \cos \frac{\alpha_2}{2} - \sin \frac{\alpha_1}{2} \sin \frac{\alpha_2}{2} = \cos \frac{\alpha}{2} \]

\[ \Rightarrow \left( \cos \frac{\alpha_1}{2} \cos \frac{\alpha_2}{2} - \sin \frac{\alpha_1}{2} \sin \frac{\alpha_2}{2} \right) = \left( \cos \frac{\alpha}{2} \right)^2 \]

\[ \Rightarrow \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} + \sin^2 \frac{\alpha_1}{2} \sin^2 \frac{\alpha_2}{2} = 2 \sin \frac{\alpha_1}{2} \sin \frac{\alpha_2}{2} \cos \frac{\alpha_1}{2} \cos \frac{\alpha_2}{2} = \cos^2 \frac{\alpha}{2} \]

\[ \Rightarrow \cos^2 \frac{\alpha_1}{2} + \left( 1 - \cos^2 \frac{\alpha_1}{2} \right) \left( 1 - \cos^2 \frac{\alpha_2}{2} \right) - \cos^2 \frac{\alpha}{2} \]

\[ \Rightarrow \cos^2 \frac{\alpha_1}{2} + \cos^2 \frac{\alpha_2}{2} + 1 - \cos^2 \frac{\alpha_1}{2} - \cos^2 \frac{\alpha_2}{2} + \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} - \cos^2 \frac{\alpha}{2} = 2 \sin \frac{\alpha_1}{2} \sin \frac{\alpha_2}{2} \cos \frac{\alpha_1}{2} \cos \frac{\alpha_2}{2} \]

\[ \Rightarrow 2 \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} - \cos^2 \frac{\alpha_1}{2} - \cos^2 \frac{\alpha_2}{2} + \sin^2 \frac{\alpha}{2} = 2 \cos \frac{\alpha_1}{2} \cos \frac{\alpha_2}{2} \left( 1 - \cos^2 \frac{\alpha}{2} \right) \sqrt{\left( 1 - \cos^2 \frac{\alpha}{2} \right)} \]
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$$\Rightarrow \left(2 \cos^2 \frac{\alpha_1}{2} \cos \frac{\alpha_2}{2} - \cos^2 \frac{\alpha_3}{2} - \cos^2 \frac{\alpha_4}{2} + \sin^2 \frac{\alpha_2}{2}\right)^2 = 4 \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} \left(1 - \cos^2 \frac{\alpha_3}{2}\right) \left(1 - \cos^2 \frac{\alpha_4}{2}\right)$$

$$\Rightarrow 4 \cos^4 \frac{\alpha_1}{2} \cos^4 \frac{\alpha_2}{2} - 2 \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2}\left(1 - \cos^2 \frac{\alpha_3}{2}\right) - 2 \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2}\left(1 - \cos^2 \frac{\alpha_4}{2}\right)$$

$$+ \cos^2 \frac{\alpha_1}{2} + \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} - \cos^2 \frac{\alpha_1}{2} \sin^2 \frac{\alpha_2}{2} - 2 \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} + \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2}$$

$$+ \cos^2 \frac{\alpha_2}{2} - \cos^2 \frac{\alpha_1}{2} \sin^2 \frac{\alpha_2}{2} + 2 \cos^2 \frac{\alpha_2}{2} \cos^2 \frac{\alpha_2}{2} \sin^2 \frac{\alpha_2}{2} - \cos^2 \frac{\alpha_1}{2} \sin^2 \frac{\alpha_2}{2} - \cos^2 \frac{\alpha_2}{2} \sin^2 \frac{\alpha_2}{2}$$

$$+ \sin^2 \frac{\alpha_2}{2} = 4 \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} - 4 \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} - 4 \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} + 4 \cos^4 \frac{\alpha_1}{2} \cos^4 \frac{\alpha_2}{2}$$

$$\Rightarrow \cos \frac{\alpha_1}{2} + \cos \frac{\alpha_2}{2} + \sin \frac{\alpha_1}{2} + 4 \cos \frac{\alpha_1}{2} \cos \frac{\alpha_2}{2} \sin \frac{\alpha_2}{2} - 2 \cos \frac{\alpha_1}{2} \cos \frac{\alpha_2}{2} \sin \frac{\alpha_2}{2} = 0$$

$$\Rightarrow \cos^4 \frac{\alpha_1}{2} + \cos^4 \frac{\alpha_2}{2} + \sin^4 \frac{\alpha_1}{2} + 4 \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} \sin^2 \frac{\alpha_2}{2} - 2 \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} \sin^2 \frac{\alpha_2}{2} = 0$$

Now, substituting all the corresponding values from eq(I), (II) & (III) in above expression, we have

$$\left(\frac{R(R - b - c)}{(R - b)(R - c)}\right)^4 + \left(\frac{R(R - a - c)}{(R - a)(R - c)}\right)^4 + \left(\frac{ab}{(R - a)(R - b)}\right)^4$$

$$+ \left(\frac{R(R - b - c)}{(R - b)(R - c)}\right)^2 \left(\frac{R(R - a - c)}{(R - a)(R - c)}\right)^2 \left(\frac{ab}{(R - a)(R - b)}\right)^2 - 2 \left(\frac{ab}{(R - a)(R - b)}\right)^2 \left(\frac{R(R - a - c)}{(R - a)(R - c)}\right)^2 = 0$$

$$\Rightarrow \frac{R^2(R - b - c)^2}{(R - b)^2(R - c)^2} + \frac{R^2(R - a - c)^2}{(R - a)^2(R - c)^2} + \frac{a^2b^2}{4ab}$$

$$+ \frac{R^2(R - b - c)(R - a - c)}{(R - a)(R - b)(R - c)} \left(\frac{R(R - a)(R - b)}{R - a - c}\right)$$

$$- \frac{2abR}{(R - a)(R - b)(R - c)} \left(\frac{R - b - c}{R - b} + \frac{R - a - c}{R - a}\right) = 0$$

Now, on multiplying the above equation by \((R - a)^2(R - b)^2(R - c)^2\), we get

$$R^2(R - a)^2(R - b - c)^2 + R^2(R - b)^2(R - a - c)^2 + a^2b^2(R - c)^2$$

$$+ R^4(R - b - c)(R - a - c)(2ab - 2R^2 + 2(a + b)R)$$

$$- 2abR(R - c)((R - a)(R - b - c) + (R - b)(R - a - c)) = 0$$

$$\Rightarrow R^2(R - a)^2(R - b)^2 + (b + c)^2 + R^2(R - b)^2(R - a + c)^2 + R^2(R - a)^2(R - b + c)^2$$

$$+ a^2b^2(R^2 - 2cR + c^2)$$

$$+ R^2(R^2 - (a + b + 2c)R + (a + c)(b + c))[2ab - 2R^2 + 2(a + b)R)$$

$$- 2abR(R - c)(2R^2 - 2(a + b + c)R + a(b + c) + b(a + c)) = 0$$

$$\Rightarrow (R^4 - 2(b + c)(R)^3 + (b + c)^2R^2)(R^2 + a^2 - 2aR) + (R^4 - 2(a + c)(R)^3 + (a + c)^2R^2)(R^2 + b^2 - 2bR)$$

$$+ a^2b^2R^2 - 2a^2b^2c + a^2b^2c^2$$

$$+ (R^4 - (a + b + 2c)(R)^3 + (a + c)(b + c)R^2)[2ab - 2R^2 + 2(a + b)R)$$

$$+ (2abcR - 2abR^2)(2R^2 - 2(a + b + c)R + a(b + c) + b(a + c)) = 0$$
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\[ R^6 - 2(b + c)R^5 + (b + c)^2R^4 + a^2R^4 - 2a^2(b + c)R^3 + a^2(b + c)^2R^2 - 2ab^5 + 4a(b + c)R^4 \]
\[ - 2a(b + c)R^3 + R^6 - 2(a + c)R^5 + (a + c)^2R^4 + b^2R^4 - 2b^2(a + c)R^3 \]
\[ + b^2(a + c)^2R^2 - 2bR^5 + 4b(a + c)R^4 - 2b(a + c)^2R^3 + a^2b^2R^2 - 2a^2b^2R + a^2b^2c^2 \]
\[ + 2abR^4 - 2ab(a^2 + 2b + c)R^3 + 2ab(a + c)(b + c)R^2 - 2R^6 + 2(a + b + 2c)R^5 \]
\[ - 2(a + c)(b + c)R^3 - 4abc(a + b + c)R^2 + 4ab(a + b + c)^2R^2 + 2abc(2ab + ac + bc)R \]
\[ - 2ab(2ab + ac + bc)R^2 = 0 \]

\[ \Rightarrow \{a^2b^2 + b^2c^2 + c^2a^2 - 2abc(a + b + c)\}R^2 + 2abc(ab + bc + ca)R + a^2b^2c^2 = 0 \]

Now, solving the above quadratic equation for the values of \( R \) as follows

\[ R = \frac{-2abc(ab + bc + ca) \pm \sqrt{\{-2abc(ab + bc + ca)\}^2 - 4\{a^2b^2 + b^2c^2 + c^2a^2 - 2abc(a + b + c)\}}}{2\{a^2b^2 + b^2c^2 + c^2a^2 - 2abc(a + b + c)\}} \]
\[ = \frac{-2abc(ab + bc + ca) \pm 2abc\sqrt{4a^2bc + 4ab^2c + 4abc^2}}{2\{a^2b^2 + b^2c^2 + c^2a^2 - 2abc(a + b + c)\}} = \frac{-abc(ab + bc + ca) \pm abc\sqrt{4abc(a + b + c)}}{(ab + bc + ca)^2 - 4abc(a + b + c)} \]
\[ \Rightarrow R = abc \left( \frac{-\frac{ab + bc + ca}{(ab + bc + ca)^2} \pm \frac{2\sqrt{abc(a + b + c)}}{2\sqrt{abc(a + b + c)}}}{(ab + bc + ca)^2 - 4abc(a + b + c)} \right) \]

Case 1: Taking positive sign, we get

\[ R = \frac{abc}{(ab + bc + ca)^2 - (2\sqrt{abc(a + b + c)})^2} = \frac{1}{(ab + bc + ca)^2 + 2\sqrt{abc(a + b + c)}} \]
\[ \Rightarrow \quad R < 0 \quad \forall \ a, b, c > 0 \quad \text{but} \quad R > 0 \quad \text{hence this value of radius} \ R \ \text{is discarded} \]

Case 2: Taking negative sign, we get

\[ R = \frac{abc}{(ab + bc + ca)^2 - (2\sqrt{abc(a + b + c)})^2} = \frac{1}{(ab + bc + ca)^2 - 2\sqrt{abc(a + b + c)}} \]
\[ = \frac{1}{2\sqrt{abc(a + b + c)} - (ab + bc + ca)} \]
\[ \Rightarrow \quad R > 0 \quad \forall \ a, b, c > 0 \quad \text{hence this value of radius} \ R \ \text{is accepted} \]

Hence, the radius \( R \) of circumscribed circle is given as

\[ R = \frac{abc}{2\sqrt{abc(a + b + c)} - (ab + bc + ca)} \quad (R > 0 \quad \forall \ a, b, c > 0) \]

Above is the required expression to compute the radius \( R \) of the circumscribed circle which is internally touched by three given circles with radii \( a, b \ & \ c \) touching each other externally.

\textbf{NOTE:} The circumscribed circle will exist for three given radii \( a, b \ & \ c \ (a \geq b \geq c > 0) \) if & only if the following inequality is satisfied

\[ a^2b^2 + b^2c^2 + c^2a^2 - 2abc(a + b + c) < 0 \]
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\[ c > \frac{ab}{(\sqrt{a} + \sqrt{b})^2} \]

For any other value of radius \( c \) (of smallest circle) not satisfying the above inequality, the circumscribed circle will not exist i.e. there will be no circle which circumscribes & internally touches three externally touching circles if the above inequality fails to hold good.

**Special case:** If three circles of equal radius \( a \) are touching each other externally then the radii \( r \) & \( R \) of inscribed & circumscribed circles respectively are obtained by setting \( a = b = c = a \) in the above expressions as follows

\[ \Rightarrow r = \frac{abc}{2\sqrt{abc(a + b + c)} + (ab + bc + ca)} = \frac{a^3}{2\sqrt{a^3(a + a + a)} + (a^2 + a^2 + a^2)} = \frac{a^3}{2\sqrt{3a^2} + 3a^2} = \frac{a}{2\sqrt{\sqrt{3} + 3}} = \frac{a(2\sqrt{3} - 3)}{(2\sqrt{3} + 3)(2\sqrt{3} - 3)} = \frac{a(2\sqrt{3} - 3)}{3} = a\left(\frac{2}{\sqrt{3}} - 1\right) \approx 0.154700538a \]

\[ \Rightarrow R = \frac{abc}{2\sqrt{abc(a + b + c)} - (ab + bc + ca)} = \frac{a^3}{2\sqrt{a^3(a + a + a)} - (a^2 + a^2 + a^2)} = \frac{a^3}{2\sqrt{3a^2} - 3a^2} = \frac{a}{2\sqrt{\sqrt{3} - 3}} = \frac{a(2\sqrt{3} + 3)}{(2\sqrt{3} - 3)(2\sqrt{3} + 3)} = \frac{a(2\sqrt{3} + 3)}{3} = a\left(\frac{2}{\sqrt{3}} + 1\right) \approx 2.154700538a \]

4. Derivation of the radius of inscribed circle: Let \( r \) be the radius of inscribed circle, with centre \( C \), externally touching two given externally touching circles, having centres \( A \) & \( B \) and radii \( a \) & \( b \) respectively, and their common tangent \( MN \). Now join the centres \( A, B \) & \( C \) to each other as well as to the points of tangency \( M, N \) & \( P \) respectively by dotted straight lines. Draw the perpendicular \( AT \) from the centre \( A \) to the line \( BN \). Also draw a line passing through the centre \( C \) parallel to the tangent \( MN \) which intersects the lines \( AM \) & \( BN \) at the points \( Q \) & \( S \) respectively. (As shown in the figure 4) Thus we have

\[
AM = a, \quad BN = b, \quad CP = r =? \\
\]

In right \( \Delta ATB \)

\[ AB = a + b \ & \ BT = BN - TN = BN - AM = b - a \]

\[ \Rightarrow AT = \sqrt{(AB)^2 - (BT)^2} \]

\[ = \sqrt{(a + b)^2 - (b - a)^2} = \sqrt{4ab} = 2\sqrt{ab} \quad \therefore AT = QS = MN = 2\sqrt{ab} \quad \ldots \ldots \ldots \ldots (I) \]

In right \( \Delta AQC \)

\[ AC = a + r \ & \ AQ = AM - QM = AM - CP = a - r \]

\[ \Rightarrow QC = \sqrt{(AC)^2 - (AQ)^2} \]

\[ = \sqrt{(a + r)^2 - (a - r)^2} = \sqrt{4ar} = 2\sqrt{ar} \quad \therefore QC = MP = 2\sqrt{ar} \quad \ldots \ldots \ldots (II) \]

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In right $\Delta BSC$

$$BC = b + r \ & \ BS = BN - SN = BN - CP = b - r$$

$$\Rightarrow CS = \sqrt{(BC)^2 -(BS)^2} = \sqrt{(b + r)^2 -(b - r)^2} = \sqrt{4br} = 2\sqrt{br} \ : \ CS = PN = 2\sqrt{br} \ \cdots \cdots \ (III)$$

From the above figure 4, it is obvious that $MP + PN = MN$ now, substituting the corresponding values, we get

$$2\sqrt{ar} + 2\sqrt{br} = 2\sqrt{ab} \ \Rightarrow \ \sqrt{r}\sqrt{a + b} = \sqrt{ab} \ \Rightarrow \ \sqrt{r} = \frac{\sqrt{ab}}{\sqrt{a + b}}$$

$$\Rightarrow r = \left(\frac{\sqrt{ab}}{\sqrt{a + b}}\right)^2 = \frac{ab}{a + b + 2\sqrt{ab}} \ \therefore \ \frac{ab}{a + b + 2\sqrt{ab}} = \frac{ab}{(\sqrt{a} + \sqrt{b})^2}(r > 0 \ \forall \ a, b > 0)$$

Above is the required expression to compute the radius ($r$) of the inscribed circle which externally touches two given circles with radii $a$ & $b$ & their common tangent.

Special case: If two circles of equal radius $a$ are touching each other externally then the radius $r$ of inscribed circle externally touching them as well as their common tangent, is obtained by setting $a = b = a$ in the above expressions as follows

$$r = \frac{ab}{a + b + 2\sqrt{ab}} = \frac{a^2}{a + a + 2\sqrt{a^2}} = \frac{a^2}{4a} = \frac{a}{4} \Rightarrow r = \frac{a}{4}$$

5. Relationship of the radii of three externally touching circles enclosed in a smallest rectangle:

Consider any three externally touching circles with the centres A, B & C and their radii $a, b \ & \ c$ ($\forall a > b \geq c$) respectively enclosed in a smallest rectangle PQRS. (See the figure 5)

Now, draw the perpendiculat AD, AF & AH from the centre A of the biggest circle to the sides PQ, RS & QR respectively. Also draw the perpendiculars CE & CM from the centre C to the straight lines PQ & DF respectively and the perpendiculars BG & BN from the centre B to the straight lines RS & DF respectively. Then join the centres A, B & C to each other by the (dotted) straight lines to obtain $\Delta ABC$. Now, we have

$AD = AF = a, \ BG = b \ & \ CE = c \ (\forall b, c < a)$

$AB = a + b, \ BC = b + c \ & \ AC = a + c$

Now, applying cosine rule in right $\Delta ABC$

Figure 5: Three externally touching circles with their centres A, B & C and radii $a, b \ & \ c$ ($\forall \ a > b \geq c$) respectively are enclosed in a smallest rectangle PQRS.
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\[ \cos \measuredangle BAC = \frac{(AB)^2 + (AC)^2 - (BC)^2}{2(AB)(AC)} = \cos \alpha = \frac{(a + b)^2 + (a + c)^2 - (b + c)^2}{2(a + b)(a + c)} \]

\[ \cos \alpha = \frac{a^2 + b^2 + 2ab + a^2 + c^2 + 2ac - b^2 - c^2 - 2bc}{2(a + b)(a + c)} = \frac{a^2 + ab + ac - bc}{(a + b)(a + c)} = \frac{a(a + b) + c(a - b)}{(a + b)(a + c)} \]

\[ \Rightarrow \sin \alpha = \sqrt{1 - \cos^2 \alpha} = \sqrt{1 - \left(\frac{a(a + b) + c(a - b)}{(a + b)(a + c)}\right)^2} \]

\[ = \frac{\sqrt{a^2(a + b)^2 + c^2(a + b)^2 + 2ac(a + b)^2 - a^2(a + b)^2 - c^2(a - b)^2 - 2ac(a^2 - b^2)}}{(a + b)(a + c)} \]

\[ \sin \alpha = \frac{\sqrt{4abc^2 + 4abc(a + b)}}{(a + b)(a + c)} \]

In right \( \triangle ANB \)

\[ \sin \measuredangle ABN = \frac{AN}{AB} = \frac{AF - NF}{AB} = \frac{AF - BG}{AB} \]

\[ \Rightarrow \sin \theta = \frac{a - b}{a + b} \]

In right \( \triangle AMC \)

\[ \sin \measuredangle ACM = \frac{AM}{AC} = \frac{AD - MD}{AC} = \frac{AD - CE}{AC} \Rightarrow \sin(\alpha - \theta) = \frac{a - c}{a + c} \]

\[ \Rightarrow (a + c)\sin(\alpha - \theta) = a - c \text{ or } (a + c)(\sin \alpha \cos \theta - \cos \alpha \sin \theta) = a - c \]

Now, by substituting the corresponding values from the eq(I), (II), (III) & (IV) in the above expression, we get

\[ (a + c) \left( \frac{\sqrt{4abc^2 + 4abc(a + b)}}{(a + b)(a + c)} \times \frac{2\sqrt{ab}}{a + b} - \frac{a(a + b) + c(a - b)}{(a + b)(a + c)} \times \frac{a - b}{a + b} \right) = a - c \]

\[ \Rightarrow (a + c) \left( \frac{4ab\sqrt{c^2 + c(a + b)}}{(a + b)^2(a + c)} - \frac{a(a + b)(a - b) + c(a - b)^2}{(a + b)^2(a + c)} \right) = a - c \]

\[ \Rightarrow 4ab\sqrt{c^2 + c(a + b)} - a(a^2 - b^2) - c(a - b)^2 = (a - c)(a + b)^2 \]

\[ \Rightarrow 4ab\sqrt{c^2 + c(a + b)} = a(a + b)^2 - c(a + b)^2 + a(a^2 - b^2) + c(a - b)^2 \]

\[ \Rightarrow 4ab\sqrt{c^2 + c(a + b)} = a((a + b)^2 + a^2 - b^2) - c((a + b)^2 - (a - b)^2) \]

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\[ 4ab\sqrt{c^2 + c(a + b)} = a(2a^2 + 2ab) - c(4ab) \]
\[ 2b\sqrt{c^2 + c(a + b)} = a(a + b) - 2bc \]

Now, taking the square on both the sides, we get

\[ (2b\sqrt{c^2 + c(a + b)})^2 = (a(a + b) - 2bc)^2 \]
\[ 4b^2(c^2 + c(a + b)) = a^2(a + b)^2 + 4b^2c^2 - 4abc(a + b) \]
\[ 4b^2c^2 + 4b^2c(a + b) = a^2(a + b)^2 + 4b^2c^2 - 4abc(a + b) \]
\[ 4b^2c(a + b) + 4abc(a + b) = a^2(a + b)^2 \]
\[ 4bc(a + b)(b + a) = a^2(a + b)^2 \text{ or } 4b^2c(a + b)^2 = a^2(a + b)^2 \Rightarrow 4bc = a^2 \]

\[ a^2 = 4bc \text{ or } a = 2\sqrt{bc} \quad \forall a > b \geq c \]

Above relation is very important for computing any of the radii \(a, b \& c\) if other two are known for three externally touching circles enclosed in a smallest rectangle.

Dimensions of the smallest enclosing rectangle: The length \(L\) & width \(B\) of the smallest rectangle \(PQRS\) enclosing three externally touching circles are calculated as follows (see the figure 5 above)

\[ \text{Length, } L = PQ = RS = SF + FG + GR = a + 2\sqrt{ab} + b = a + b + 2\sqrt{ab} = (\sqrt{a} + \sqrt{b})^2 \]
\[ \text{Width, } B = PS = QR = DM = 2a \]

\[ \therefore \text{Length}, L = (\sqrt{a} + \sqrt{b})^2 \quad \text{& Width}, B = 2a \quad \forall a^2 = 4bc \quad \& \ a > b \geq c \]

Thus, above expressions can be used to compute the dimensions of the smallest rectangle enclosing three externally touching circles having radii \(a, b \& c\) \((a > b \geq c)\).

6. Length of common chord of two intersecting circles: Consider two circles with centres \(O_1\) & \(O_2\) and radii \(r_1\) & \(r_2\) respectively, at a distance \(d\) between their centres, intersecting each other at the points \(A \& B\) (As shown in the figure 6). Join the centres \(O_1\) & \(O_2\) to the point \(A\). The line \(O_1O_2\) bisects the common chord \(AB\) perpendicularly at the point \(M\). Let \(AM = x\) then the length of common chord \(AB = 2x\). Now

In right triangle \(\triangle AMO_1\),

\[ O_1M = \sqrt{(O_1A)^2 - (AM)^2} = \sqrt{r_1^2 - x^2} \]

Similarly, in right triangle \(\triangle AMO_2\),

\[ MO_2 = \sqrt{(O_2A)^2 - (AM)^2} = \sqrt{r_2^2 - x^2} \]

Now,

\[ O_1O_2 = O_1M + MO_2 \]

Figure 6: Two circles with the centres \(O_1\) & \(O_2\) and radii \(r_1\) & \(r_2\) respectively at a distance \(d\) between their centres, intersecting each other at the points \(A \& B\)
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Substituting the corresponding values, we get

\[ d = \sqrt{r_1^2 - x^2} + \sqrt{r_2^2 - x^2} \]

Taking squares on both the sides,

\[ d^2 = \left(\sqrt{r_1^2 - x^2} + \sqrt{r_2^2 - x^2}\right)^2 \]

\[ r_1^2 - x^2 + r_2^2 - x^2 + 2\sqrt{(r_1^2 - x^2)(r_2^2 - x^2)} = d^2 \]

\[ 2\sqrt{(r_1^2 - x^2)(r_2^2 - x^2)} = 2x^2 + d^2 - r_1^2 - r_2^2 \]

\[ 4(r_1^2 - x^2)(r_2^2 - x^2) = (2x^2 + d^2 - r_1^2 - r_2^2)^2 \]

\[ 4r_1^2r_2^2 - 4(r_1^2 + r_2^2)x^2 + 4x^4 = 4x^4 + (d^2 - r_1^2 - r_2^2)^2 + 4(d^2 - r_1^2 - r_2^2)x^2 \]

\[ 4d^2x^2 = 4r_1^2r_2^2 - (d^2 - r_1^2 - r_2^2)^2 \]

\[ 4x^2 = \frac{(2r_1r_2)^2 - (d^2 - r_1^2 - r_2^2)^2}{d^2} \]

\[ 4x^2 = \frac{(2r_1r_2 + d^2 - r_1^2 - r_2^2)(2r_1r_2 - d^2 + r_1^2 + r_2^2)}{d^2} \]

\[ 4x^2 = \frac{(d^2 - (r_1 - r_2)^2)((r_1 + r_2)^2 - d^2)}{d^2} \]

\[ 2x = \sqrt{\frac{(d^2 - (r_1 - r_2)^2)((r_1 + r_2)^2 - d^2)}{d^2}} \]

\[ \Rightarrow AB = \frac{\sqrt{(d^2 - (r_1 - r_2)^2)((r_1 + r_2)^2 - d^2)}}{d} \]

Hence, the length of the common chord of two intersecting circles with radii \( r_1 \) & \( r_2 \) at a distance \( d \) between their centres is

**Length of common chord, \( L = \frac{\sqrt{(d^2 - (r_1 - r_2)^2)((r_1 + r_2)^2 - d^2)}}{d} \quad \forall \ r_1 - r_2 \leq d \leq r_1 + r_2**

**Special case:** If \( r_1 \neq r_2 \) then the **maximum length of common chord** of two intersecting circles

\[ = 2 \times \min(r_1, r_2) = \text{diameter of smaller circle} \text{ at a central distance } d = \sqrt{|r_1^2 - r_2^2|} \]

**Angles of intersection of two intersecting circles:** Let \( \angle AO_1M = \theta_1 \) & \( \angle AO_2M = \theta_2 \) (See above fig 6).

In right \( \triangle AMO_1 \), we have

\[ \sin \angle AO_1M = \frac{AM}{AO_1} \quad \Rightarrow \quad \sin \theta_1 = \frac{L/2}{r_1} = \frac{L}{2r_1} \]

\[ \Rightarrow \theta_1 = \sin^{-1} \left( \frac{L}{2r_1} \right) = \text{semi aperture angle subtended by common chord } AB \text{ at centre } O_1 \]

Similarly, in right \( \triangle AMO_2 \),
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\[ \angle O_2 M = \theta_2 = \sin^{-1}\left(\frac{L}{2r_2}\right) \]

semi aperture angle subtended by common chord AB at centre \( O_2 \)

Now, it can be easily proved that one of the two supplementary angles of intersection (\( \theta \)) is given as the sum of above two semi-aperture angles \( \theta_1 \) and \( \theta_2 \) subtended by common chord AB at the centres \( O_1 \) & \( O_2 \) of two intersecting circles (see above fig. 6)

\[ \theta = \theta_1 + \theta_2 = \sin^{-1}\left(\frac{L}{2r_1}\right) + \sin^{-1}\left(\frac{L}{2r_2}\right) \]

Hence, both the \textit{supplementary angles of intersection of two intersecting circles} are given as follows

\[ \theta = \sin^{-1}\left(\frac{L}{2r_1}\right) + \sin^{-1}\left(\frac{L}{2r_2}\right) \quad \& \quad \pi - \theta \]

Where, \[ L = \sqrt{\frac{(d^2 - (r_1 - r_2)^2)(r_1 + r_2)^2 - d^2}{d}} \quad \forall \quad |r_1 - r_2| \leq d \leq r_1 + r_2 \]

**Area of intersection (A) of two intersecting circles:** As we have computed above, \( 2\theta_1 \) is the angle of aperture subtended by common chord AB at the centre \( O_1 \) of circle with a radius \( r_1 \) hence the area \( (A_1) \) of segment of corresponding circle is give as (Refer to fig.6 above)

\[
A_1 = \text{Area of sector } O_1 AB - \text{area of isosceles } \Delta O_1 AB \\
= \frac{1}{2}(2\theta_1)r_1^2 - \frac{1}{2}(r_1 \times r_1) \sin 2\theta_1 \\
= \frac{1}{2}(2\theta_1 - \sin 2\theta_1)r_1^2 \\
= \frac{1}{2}(2\theta_1 - 2 \sin \theta_1 \cos \theta_1)r_1^2 \\
= (\theta_1 - \sin \theta_1 \cos \theta_1)r_1^2
\]

Similarly, the area \( (A_2) \) of segment of circle with a radius \( r_2 \) & aperture angle \( 2\theta_2 \) subtended by common chord AB at the centre \( O_2 \), is given as follows

\[
A_2 = (\theta_2 - \sin \theta_2 \cos \theta_2)r_2^2
\]

Now, the area of intersection (A) of two intersecting circles will be equal to the sum of areas \( A_1 \) & \( A_2 \) of segments as computed above

\[
A = A_1 + A_2 = (\theta_1 - \sin \theta_1 \cos \theta_1)r_1^2 + (\theta_2 - \sin \theta_2 \cos \theta_2)r_2^2
\]

Hence, the \textit{area (A)} of intersection of any two intersecting circles of radii \( r_1 \) & \( r_2 \) separated by a central distance \( d \) is given as

\[
A = (\theta_1 - \sin \theta_1 \cos \theta_1)r_1^2 + (\theta_2 - \sin \theta_2 \cos \theta_2)r_2^2
\]

Where, \[ \theta_1 = \sin^{-1}\left(\frac{L}{2r_1}\right), \quad \theta_2 = \sin^{-1}\left(\frac{L}{2r_2}\right) \quad \& \quad L = \sqrt{\frac{(d^2 - (r_1 - r_2)^2)(r_1 + r_2)^2 - d^2}{d}} \quad \forall \quad |r_1 - r_2| \leq d \leq r_1 + r_2
\]

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Conclusion: All the articles above have been derived by using simple geometry & trigonometry. All above articles (formula) are very practical & simple to apply in case studies & practical applications of 2-D Geometry. Although above results are also valid in case of three spheres touching one another externally in 3-D geometry.

Note: Above articles had been derived & illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)

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