**Mathematical Analysis of Truncated Rhombic Dodecahedron**

*(Application of HCR’s Theory of Polygon)*

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**Introduction:** A truncated rhombic dodecahedron is a non-uniform convex polyhedron which has 12 congruent rectangular faces, 6 congruent square faces & 8 congruent equilateral-triangular faces, 48 edges (24 small & 24 large edges) & 24 identical vertices (at each of which two rectangular, one square & one triangular faces meet) lying on a spherical surface of certain radius. It is derived by truncating all 14 identical vertices of a rhombic dodecahedron at the points of tangency (where an inscribed circle touches four sides of rhombic face) such that its every rhombic face is changed into a rectangular face which has its length $\sqrt{2}$ times the width and all its 24 edges are changed into 24 new identical vertices (as shown in fig 1). This truncated rhombic dodecahedron looks very similar to a rhombicuboctahedron but a truncated rhombic dodecahedron has 12 rectangular faces instead of square faces. The number of faces, edges & vertices of a truncated rhombic dodecahedron generated by truncating all the vertices of parent polyhedron (i.e. rhombic dodecahedron) are obtained as follows

Number of new rectangular faces = number of faces in parent solid (rhombic dodecahedron) = 12

Number of new square faces = number of vertices where 4 edges meet in parent solid = 6

Number of new triangular faces = number of vertices where 3 edges meet in parent solid = 8

Number of new edges = (no. of edges in rhombus)\(\times\)(no. of rhombic faces in parent solid) = 4 \cdot 12 = 48

Number of new vertices = number of edges in parent solid = 24

Since a truncated rhombic dodecahedron is derived from a rhombic dodecahedron hence it becomes much easier and simpler to mathematically analyse & derive analytic formula for a truncated rhombic dodecahedron by using the mathematical relations and formula derived for a rhombic dodecahedron (refer to ‘Mathematical analysis of rhombic dodecahedron’ for derivations/formula). For this we will consider a parent rhombic dodecahedron of edge length $a$ (which is truncated at the points of tangency of inscribed circle & rhombic face as shown in fig 1 above) to derive a truncated rhombic dodecahedron of small & large edge lengths & $s\sqrt{2}$ respectively. We will establish a mathematical relation between edge length $a$ of parent polyhedron (i.e. rhombic dodecahedron) & small edge length $s$ of truncated rhombic dodecahedron. Thus we will derive the formula to analytically compute the radius of circumscribed sphere passing through all 24 identical vertices, normal distances of rectangular, square & equilateral triangular faces from the centre of polyhedron, surface area, volume, solid angles subtended by rectangular, square & equilateral triangular faces at the centre of polyhedron by using ‘HCR’s Theory of Polygon’, dihedral angle between each two faces meeting at any of 24 identical vertices (i.e. truncated rhombic dodecahedron), solid angle subtended by truncated rhombic dodecahedron at any of its 24 identical vertices.
Derivation of radius \( R \) of circumscribed sphere i.e. passing through all 24 identical vertices of a truncated rhombic dodecahedron: Consider a rhombic dodecahedron having 12 congruent faces each as a rhombus of side \( a \). We know that the midsphere with radius \( R_{md} \) touches all 24 edges of rhombic dodecahedron i.e. midsphere touches all four sides of each rhombic face. Consider a rhombic face \( ABCD \) touching the midsphere at four distinct points (of tangency) \( E, F, G \) & \( I \) on the sides \( AB, BC, CD \) & \( AD \) respectively. Circle passing through the points of tangency \( E, F, G \) & \( I \) is inscribed by the rhombus \( ABCD \) & lies on the surface of midsphere of rhombic dodecahedron. Join these four points of tangency \( E, F, G \) & \( I \) to get a rectangular face \( EFGI \) (as shown in fig-2). Thus we mark a rectangle on each of 12 congruent rhombic faces by joining the points of tangency of the edges & the midsphere and then truncate the rhombic dodecahedron from each of its 14 vertices at the point of tangency to get a truncated rhombic dodecahedron (as shown in above fig-1). From formula derived in 'Mathematical analysis of rhombic dodecahedron', the angles \( \alpha \) & \( \beta \), the lengths of semi major & semi minor diagonals \( AM \) & \( BM \) of rhombic face \( ABCD \), and the radius \( R_{md} \) of midsphere of rhombic dodecahedron with edge length \( a \), are given as

\[
\begin{align*}
\alpha &= 2 \cot^{-1} \sqrt{2}, \quad \beta = 2 \tan^{-1} \sqrt{2}, \quad AM = a \sqrt{\frac{2}{3}}, \quad BM = \frac{a}{\sqrt{3}}, \quad R_{md} = \frac{2a\sqrt{2}}{3}
\end{align*}
\]

In right \( \Delta AEM \) (see fig-2)

\[
\cos \frac{\alpha}{2} = \frac{AE}{AM} \Rightarrow AE = AM \cos \frac{\alpha}{2} = a \sqrt{\frac{2}{3}} \cos \left( \frac{2 \cot^{-1} \sqrt{2}}{2} \right) = a \sqrt{\frac{2}{3}} \cos \left( \frac{2}{\sqrt{2}} \right) = a \sqrt{\frac{2}{3}} \left( \frac{2}{2} \right) = \frac{2a}{3}
\]

\[
\Rightarrow BE = AB - AE = a - \frac{2a}{3} = \frac{a}{3}
\]

In right \( \Delta ANE \) (see fig-2)

\[
\sin \frac{\alpha}{2} = \frac{EN}{AE} \Rightarrow EN = AE \sin \frac{\alpha}{2} = \frac{2a}{3} \sin \left( \frac{2 \cot^{-1} \sqrt{2}}{2} \right) = \frac{2a}{3} \sin \left( \sin^{-1} \frac{1}{\sqrt{3}} \right) = \frac{2a}{3} \left( \frac{1}{\sqrt{3}} \right) = \frac{2a}{3 \sqrt{3}}
\]

\[
\Rightarrow EI = 2EN = 2 \left( \frac{2a}{3 \sqrt{3}} \right) = \frac{4a}{3 \sqrt{3}} \quad \ldots \ldots \quad (I)
\]

Similarly, in right \( \Delta BPE \) (see fig-2 above)

\[
\sin \frac{\beta}{2} = \frac{EP}{BE} \Rightarrow EP = BE \sin \frac{\beta}{2} = \frac{a}{3} \sin \left( \frac{2 \tan^{-1} \sqrt{2}}{2} \right) = \frac{a}{3} \sin \left( \sin^{-1} \frac{2}{\sqrt{3}} \right) = \frac{a}{3} \left( \frac{2}{\sqrt{3}} \right) = \frac{a}{3} \sqrt{3}
\]

\[
\Rightarrow EF = 2EP = 2 \left( \frac{a}{3} \sqrt{3} \right) = \frac{2a}{3} \sqrt{3} \quad \ldots \ldots \quad (II)
\]

Now, diving the eq. (I) by eq.(II) as follows
If the small side or width EF of the rectangular face EFGI is $s$ i.e. $EF = s$ then the large side or length EI of the rectangular face EFGI is $= EF \sqrt{2} = s \sqrt{2}$. Thus, two unequal edges i.e. length and width of each of 12 congruent rectangular faces of the truncated rhombic dodecahedron are $& s \sqrt{2}$. Now, substituting $EF = s$ in eq.(2) as follows

$$EF = s = \frac{2a}{3} \sqrt{\frac{2}{3}} \Rightarrow a = \frac{3s}{2} \sqrt{\frac{3}{2}} \ldots \ldots (III)$$

Now, the radius $R$ of the circumscribed sphere i.e. passing through all 24 identical vertices of the truncated rhombic dodecahedron with unequal edge lengths $s & s \sqrt{2}$, is given as follows

$$R = \text{Radius } R_{ma} \text{ of midsphere of parent rhombic dodecahedron with edge length } a$$

$$R = \frac{2a \sqrt{2}}{3} \text{ (from formula derived for a rhombic dodecahedron)}$$

$$R = \frac{2}{3} \left( \frac{3s}{2} \sqrt{\frac{3}{2}} \right) \sqrt{2} = s \sqrt{3} \text{ (Substituting value of } a \text{ in terms of } s \text{ from eq.(III))}$$

$$R = s \sqrt{3} = 1.732050808 \ldots \ldots (1)$$

**Normal distances $H_R, H_S & H_T$ of rectangular, square & equilateral triangular faces from the centre of a truncated rhombic dodecahedron:** Consider a rectangular face EFGI of length and width $s & s \sqrt{2}$ which is at a normal distance $OM = H_R$ from the centre O of truncated rhombic dodecahedron. Join its vertices E, F, G & I & centre M to the centre O (as shown in fig-3)

In right $\triangle OMG$ (see fig-3), using Pythagorean theorem, we get

$$OM = \sqrt{(OG)^2 - (MG)^2} = \sqrt{(R)^2 - \left(\frac{EG}{2}\right)^2} = \sqrt{\left(s \sqrt{3}\right)^2 - \left(\frac{s^2}{2} + \frac{(s \sqrt{2})^2}{2}\right)^2}$$

$$H_R = \sqrt{3s^2 - \frac{3s^2}{4}} = \sqrt{\frac{9s^2}{4}} = \frac{3s}{2}$$

Consider a square face EKLI of side $s \sqrt{2}$ which is at a normal distance $OQ = H_S$ from the centre O of truncated rhombic dodecahedron. Join its vertices E, K, L & I & centre Q to the centre O (as shown in fig-4 below)

In right $\triangle OQL$ (see fig-4 below), using Pythagorean theorem, we get

$$OQ = \sqrt{(OL)^2 - (QL)^2}$$
Consider an equilateral triangular face EFJ of side $s$ which is at a normal distance $OT = H_T$ from the centre $O$ of truncated rhombic dodecahedron. Join its vertices $E$, $F$, $J$ & centre $M$ to the centre $O$ of truncated rhombic dodecahedron (as shown in fig-5)

In right $\triangle OTF$ (see fig-5), using Pythagorean theorem, we get

$$OT = \sqrt{(OF)^2 - (TF)^2}$$

$$H_T = \sqrt{(R)^2 - \left(\frac{s}{\sqrt{3}}\right)^2}$$

(Circum radius of equilateral $\triangle = \frac{\text{side}}{\sqrt{3}}$)

$$H_T = \sqrt{\left(\frac{s}{\sqrt{3}}\right)^2 - \frac{s^2}{3}}$$

$$H_T = \sqrt{3s^2 - \frac{s^2}{3}} = \frac{\sqrt{8s^2}}{\sqrt{3}} = 2s \frac{2}{\sqrt{3}}$$

Hence, the normal distances $H_R$, $H_S$ & $H_T$ of rectangular, square & equilateral triangular faces respectively from the centre of truncated rhombic dodecahedron with unequal edges $s$ & $s\sqrt{2}$, are given as follows

$$H_R = \frac{3s}{2}, \quad H_S = s\sqrt{2} \approx 1.414213562s \quad \text{and} \quad H_T = 2s \frac{2}{\sqrt{3}} \approx 1.632993162s$$

It is clear from above values of normal distances that the equilateral triangular faces are the farthest from the centre while square faces are the closest to the centre & rectangular faces are at a normal distance between these two. For finite value of small edge length $s \Rightarrow H_S < H_R < H_T < R$
Surface Area ($A_s$) of truncated rhombic dodecahedron: The surface of a truncated rhombic dodecahedron consists of 12 congruent rectangular faces each of length $s\sqrt{2}$ & width $s$, 6 congruent square faces each with side $s\sqrt{2}$ and 8 congruent equilateral triangular faces each with side $s$. Therefore, the (total) surface area of truncated rhombic dodecahedron is given as follows

\[
A_s = 12(\text{Area of rectangular face}) + 6(\text{Area of square face}) + 8(\text{Area of equi. triangular face})
\]

\[
= 12(s \cdot s\sqrt{2}) + 6((s\sqrt{2})^2) + 8\left(\frac{\sqrt{3}}{4}(s)^2\right)
\]

\[
= 12s^2\sqrt{2} + 12s^2 + 2s^2\sqrt{3}
\]

\[
= 2s^2(6\sqrt{2} + 6 + \sqrt{3})
\]

\[
\therefore \text{Surface area, } A_s = 2s^2(6\sqrt{2} + 6 + \sqrt{3}) \approx 32.43466436 \, s^2
\]

Volume ($V$) of truncated rhombic dodecahedron: The surface of a truncated rhombic dodecahedron consists of 12 congruent rectangular faces each of length $s\sqrt{2}$ & width $s$, 6 congruent square faces each with side $s\sqrt{2}$ and 8 congruent equilateral triangular faces each with side $s$. Thus a solid truncated rhombic dodecahedron can be assumed to consisting of 12 congruent right pyramids with rectangular base of length $s\sqrt{2}$ & width $s$ & normal height $H_r$, 6 congruent right pyramids with square base of side $s\sqrt{2}$ & normal height $H_s$ and 8 congruent right pyramids with equilateral triangular base with side $s\sqrt{2}$ & normal height $H_T$ (as shown in above fig-1). Therefore, the volume of truncated rhombic dodecahedron is given as

\[
V = 12(\text{Volume of rectangular right pyramid}) + 6(\text{Volume of square right pyramid}) + 8(\text{Volume of equilateral triangular right pyramid})
\]

\[
= 12\left(\frac{1}{3} (s \cdot s\sqrt{2}) \cdot \frac{3s}{2}\right) + 6(\frac{1}{3} (s\sqrt{2})^2 \cdot s\sqrt{2}) + 8\left(\frac{1}{3} \left(\frac{\sqrt{3}}{4} s^2\right) \cdot 2s \cdot \frac{2}{\sqrt{3}}\right)
\]

\[
= 12\left(\frac{s^3}{\sqrt{2}}\right) + 6\left(\frac{2s^3\sqrt{2}}{3}\right) + 8\left(\frac{s^3}{3\sqrt{2}}\right)
\]

\[
= \frac{6s^3\sqrt{2} + 4s^3\sqrt{2} + 4s^3\sqrt{2}}{3}
\]

\[
= \frac{34s^3\sqrt{2}}{3}
\]

\[
\therefore \text{Volume, } V = \frac{34s^3\sqrt{2}}{3} \approx 16.02775371 \, s^3
\]

Mean radius ($R_m$) of truncated rhombic dodecahedron: It is the radius of the sphere having a volume equal to that of a given truncated rhombic dodecahedron with edge lengths $s$ & $s\sqrt{2}$. It is computed as follows

Volume of sphere with mean radius $R_m = \text{volume of truncated rhombic dodecahedron}$

\[
\frac{4}{3} \pi (R_m)^3 = \frac{34s^3\sqrt{2}}{3}
\]

\[
(R_m)^3 = \frac{17s^3}{\pi \sqrt{2}}
\]
Dihedral angle between any two faces meeting at a vertex of truncated rhombic dodecahedron:

A truncated rhombic dodecahedron has 24 identical vertices at each of which two rectangular, one square & one equilateral triangular faces meet. We will consider each pair of two faces meeting at a vertex & find the dihedral angle between them.

Consider a rectangular and a square faces meeting each other at the vertex E which are inclined at a dihedral angle $\theta_{RS}$ (as shown in the fig-6). The line EF shows the small side of rectangular face EFGI (see fig-3 above) & line EK shows the side of square face EKLI (see fig-4 above). Drop the perpendiculars $OM_1$ & $OQ_1$ from the centre O to the mid points $M_1$ & $Q_1$ of the sides EF & EK respectively.

In right $\triangle OM_1E$ (see fig-6),

\[
\tan \angle M_1EO = \frac{OM_1}{EM_1} = \frac{HR}{\left(\frac{EF}{2}\right)} = \frac{3s}{\left(\frac{s}{2}\right)} = 3
\]

(since, $EF = s$)

\[
\angle M_1EO = \tan^{-1} 3
\]

In right $\triangle OQ_1E$ (see fig-6),

\[
\tan \angle Q_1EO = \frac{OQ_1}{EQ_1} = \frac{HS}{\left(\frac{EK}{2}\right)} = \frac{s\sqrt{2}}{\left(\frac{s\sqrt{2}}{2}\right)} = 2
\]

(since, $EK = s\sqrt{2}$)

\[
\angle Q_1EO = \tan^{-1} 2
\]

\[
\Rightarrow \angle FEK = \angle M_1EO + \angle Q_1EO
\]

\[
\theta_{RS} = \tan^{-1} 3 + \tan^{-1} 2
\]

\[
= \pi + \tan^{-1} \left(\frac{3 + 2}{1 - 3 \cdot 2}\right) = \pi + \tan^{-1} \left(\frac{x + y}{1 - xy}\right) \forall xy > 1
\]

\[
= \pi + \left(-\frac{\pi}{4}\right)
\]

\[
= \frac{3\pi}{4}
\]
Hence, the dihedral angle $\theta_{RS}$ between any two adjacent rectangular & square faces of a truncated rhombic dodecahedron, is given as follows

$$\theta_{RS} = \frac{3\pi}{4} = 135^\circ \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (4)$$

Consider a rectangular and an equilateral triangular faces meeting each other at the vertex $E$ which are inclined at a dihedral angle $\theta_{RT}$ (as shown in the fig-7). The line $E_I$ shows the large side of rectangular face $EFGI$ (see fig-3 above) & line $EJ_1$ shows the altitude from vertex $E$ to the side $FJ$ of equilateral triangular face $EFJ$ (see fig-5 above). Drop the perpendiculares $OM_1$ & $OT_1$ from the centre $O$ to the points $M_1$ & $T_1$ of side $EI$ & altitude $EJ_1$ respectively.

In right $\Delta O M_1 E$ (see fig-7),

$$\tan \angle M_1 E O = \frac{OM_1}{EM_1} = \frac{H_E}{(E I)} = \frac{3s}{2 \sqrt{2}} = \frac{3}{\sqrt{2}} \quad \text{(since, } EI = s \sqrt{2})$$

$$\angle M_1 E O = \tan^{-1} \frac{3}{\sqrt{2}}$$

In right $\Delta O T_1 E$ (see fig-7),

$$\tan \angle T_1 E O = \frac{O T_1}{E T_1} = \frac{H_E}{(E I)} = \frac{4 \sqrt{2}}{(s \sqrt{2})} = \frac{4 \sqrt{2}}{2} \quad \text{(since, } EJ_1 = s \sin 60^\circ = \frac{s \sqrt{3}}{2})$$

$$\angle T_1 E O = \tan^{-1} 4 \sqrt{2}$$

$$\Rightarrow \angle EJ_1 = \angle M_1 E O + \angle T_1 E O$$

$$\theta_{RT} = \tan^{-1} \frac{3}{\sqrt{2}} + \tan^{-1} 4 \sqrt{2}$$

$$= \pi + \tan^{-1} \frac{3 \sqrt{2} + 4 \sqrt{2}}{1 - \frac{3}{\sqrt{2}} \cdot 4 \sqrt{2}} \quad \text{(tan^{-1} x + tan^{-1} y = \pi + tan^{-1} (\frac{x+y}{1-xy}) \forall xy > 1)}$$

$$= \pi + \tan^{-1} \frac{1}{\sqrt{2}}$$

$$= \pi - \tan^{-1} \frac{1}{\sqrt{2}}$$

Hence, the dihedral angle $\theta_{RT}$ between any two adjacent rectangular & equilateral triangular faces of a truncated rhombic dodecahedron, is given as follows

$$\theta_{RT} = \pi - \tan^{-1} \frac{1}{\sqrt{2}} \approx 144^\circ 44' 8.2'' \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (5)$$
Consider a square and an equilateral triangular faces meeting each other at the vertex $E$ which are inclined at a dihedral angle $\theta_{ST}$ (as shown in the fig-8). The line $EL$ shows the diagonal of square face $EKLI$ (see fig-4 above) & line $EF_1$ shows the altitude from vertex $E$ to the side $FJ$ of equilateral triangular face $EFJ$ (see fig-5 above). Drop the perpendiculars $OQ$ & $OT'$ from the centre $O$ to the centres $Q$ & $T$ of square & triangular faces respectively.

In right $\triangle OQE$ (see fig-8),

$$\tan \angle QEO = \frac{OQ}{QE} = \frac{H_S}{EL} = \frac{s\sqrt{2}}{2} = \sqrt{2} \quad (\text{since, } EL = \sqrt{2}(s\sqrt{2}) = 2s)$$

$\angle QEO = \tan^{-1}\sqrt{2}$

In right $\triangle OTE$ (see fig-8),

$$\tan \angle TEO = \frac{OT}{TE} = \frac{H_T}{\frac{2}{3}EF_1} = \frac{2\sqrt{3}}{2} \quad (\text{since, } EF_1 = s \sin 60^\circ = \frac{s\sqrt{3}}{2})$$

$\angle TEO = \tan^{-1}2\sqrt{2}$

$$\Rightarrow \angle LEF_1 = \angle QEO + \angle TEO$$

$$\theta_{ST} = \tan^{-1}\sqrt{2} + \tan^{-1}2\sqrt{2}$$

$$\pi + \tan^{-1}\left(\frac{\sqrt{2} + 2\sqrt{2}}{1 - \sqrt{2} \cdot 2\sqrt{2}}\right)$$

$$\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\frac{x + y}{1 - xy}\right) \forall xy > 1$$

$$\pi + \tan^{-1}(-\sqrt{2})$$

$$\pi - \tan^{-1}\sqrt{2}$$

Hence, the dihedral angle $\theta_{ST}$ between square & equilateral triangular faces meeting at the same vertex of truncated rhombic dodecahedron, is given as follows

$$\theta_{ST} = \pi - \tan^{-1}\sqrt{2} \approx 125^\circ15'51.8'' \quad ........ \quad (6)$$

Consider two congruent rectangular faces meeting each other at the vertex $E$ which are inclined at a dihedral angle $\theta_{RR}$ (as shown in the fig-9). The line $EG$ shows the diagonal of rectangular face $EFGI$ (see fig-3 above) & line $EG'$ shows the diagonal of another rectangular face $EFG'I'$. Drop the perpendiculars $OM$ & $OM'$ from the centre $O$ to the centres $M$ & $M'$ of the rectangular faces.

In right $\triangle OME$ (see fig-9),

$$\tan \angle MEO = \frac{OM}{ME} = \frac{H_R}{\frac{2}{2}} = \frac{3s}{2} = \sqrt{3} \quad \left(\text{EG} = \sqrt{s^2 + (s\sqrt{2})^2} = s\sqrt{3}\right)$$

$\angle MEO = \tan^{-1}\sqrt{3} \Rightarrow \angle M'E'O = \angle MEO = \tan^{-1}\sqrt{3} = \pi/3$
\[ \angle G'EG = \angle M'EO + \angle MEO \]

\[ \theta_{RR} = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3} \]

Hence, the dihedral angle \( \theta_{RR} \) between any two rectangular faces meeting at the same vertex of truncated rhombic dodecahedron, is given as follows

\[ \theta_{RR} = \frac{2\pi}{3} = 120^\circ \] \hspace{1cm} (7)

Solid angles \( \omega_R, \omega_S \) & \( \omega_T \) subtended by rectangular, square & equilateral triangular faces respectively at the centre of truncated rhombic dodecahedron:

We know that the solid angle (\( \omega \)), subtended by a rectangular plane of length \( l \) & width \( b \) at any point lying at a distance \( h \) on the perpendicular axis passing through the centre, is given by "HCR’s Theory of Polygon" as follows

\[ \omega = 4 \sin^{-1} \left( \frac{lb}{\sqrt{(l^2 + 4h^2)(b^2 + 4h^2)}} \right) \]

Substituting the corresponding values in above formula i.e. length \( l = s\sqrt{2} \), width \( b = s \) & normal height \( h = H_R = 3s/2 \), the solid angle \( \omega_R \) subtended by the rectangular face EFGI at the centre O of truncated rhombic dodecahedron (as shown in above fig-3), is obtained as follows

\[ \omega_R = 4 \sin^{-1} \left( \frac{s\sqrt{2} \cdot s}{\sqrt{(s\sqrt{2})^2 + 4\left(\frac{3s}{2}\right)^2}} \right) = 4 \sin^{-1} \left( \frac{s^2\sqrt{2}}{s^2\sqrt{110}} \right) = 4 \sin^{-1} \left( \frac{2}{110} \right) = 4 \sin^{-1} \left( \frac{1}{\sqrt{55}} \right) \]

Hence, the solid angle \( \omega_R \) subtended by any of 12 congruent rectangular faces at the centre of a truncated rhombic dodecahedron, is given as follows

\[ \omega_R = 4 \sin^{-1} \left( \frac{1}{\sqrt{55}} \right) \approx 0.541007833 \text{ sr} \] \hspace{1cm} (8)

Similarly, substituting the corresponding values in above formula i.e. length \( l = s\sqrt{2} \), width \( b = s\sqrt{2} \) & normal height \( h = H_S = s\sqrt{2} \), the solid angle \( \omega_S \) subtended by the square face EKLI at the centre O of truncated rhombic dodecahedron (as shown in above fig-4), is obtained as follows

\[ \omega_S = 4 \sin^{-1} \left( \frac{s\sqrt{2} \cdot s\sqrt{2}}{\sqrt{(s\sqrt{2})^2 + 4(s\sqrt{2})^2}} \right) \]
\[ \omega_s = 4 \sin^{-1}\left( \frac{2s^2}{\sqrt{(10s^2)^2 - (10s^2)^2}} \right) = 4 \sin^{-1}\left( \frac{2s^2}{10s^2} \right) = 4 \sin^{-1}(0.2) \]

Hence, the solid angle \( \omega_s \) subtended by any of 6 congruent square faces at the centre of a truncated rhombic dodecahedron, is given as follows

\[ \omega_s = 4 \sin^{-1}(0.2) \text{ sr} \approx 0.805431683 \text{ sr} \] \hspace{1cm} (9)

We know that the solid angle (\( \omega \)) subtended by any regular polygonal plane with \( n \) no. of sides each of length \( a \) at any point lying at a distance \( H \) on the vertical axis passing through the centre, is given by "HCR’s Theory of Polygon" as follows

\[ \omega = 2\pi - 2n \sin^{-1}\left( \frac{2H\sin\frac{\pi}{n}}{\sqrt{4H^2 + a^2\cot^2\frac{\pi}{n}}} \right) \]

Substituting the corresponding values in above formula i.e. number of sides \( n = 3 \) (for regular \( \Delta \)), length of each side \( a = s \), & normal height \( H = H_s = 2s\sqrt{2/3} \), the solid angle \( \omega_T \) subtended by the equilateral triangular face EFJ at the centre O of truncated rhombic dodecahedron (as shown in above fig-5), is obtained as follows

\[ \omega_T = 2\pi - 2 \times 3 \sin^{-1}\left( \frac{\sqrt{3}}{3} \right) \left( \frac{2s\sqrt{2/3}}{\sqrt{4s^2/3 + s^2/3}} \right) = 2\pi - 6 \sin^{-1}\left( \frac{2\sqrt{2/3}}{33s^2/3} \right) = 2\pi - 6 \sin^{-1}\left( \frac{2\sqrt{2}}{\sqrt{11}} \right) \]

Hence, the solid angle \( \omega_T \) subtended by any of 8 congruent equilateral triangular faces at the centre of a truncated rhombic dodecahedron, is given as follows

\[ \omega_T = 2\pi - 6 \sin^{-1}\left( \frac{2}{\sqrt{11}} \right) \text{ sr} \approx 0.155210814 \text{ sr} \] \hspace{1cm} (10)

**Total solid angle:** We know that a truncated rhombic dodecahedron has 12 congruent rectangular faces, 6 congruent square faces & 8 congruent equilateral triangular faces. Therefore, the total solid angle subtended by all the faces at the centre of a truncated rhombic dodecahedron, is given as follows

\[ \omega = 12(\omega_R) + 6(\omega_S) + 8(\omega_T) \]

\[ \omega = 12 \left( 4 \sin^{-1}\left( \frac{1}{\sqrt{55}} \right) \right) + 6 \left( 4 \sin^{-1}(0.2) \right) + 8 \left( 2\pi - 6 \sin^{-1}\left( \frac{2}{\sqrt{11}} \right) \right) \]

\[ \omega = 48 \sin^{-1}\left( \frac{1}{\sqrt{55}} \right) + 24 \sin^{-1}(0.2) + 16\pi - 48 \sin^{-1}\left( \frac{2}{\sqrt{11}} \right) \]
\[
\omega = 16\pi + 24\sin^{-1}(0.2) - 48\left(\sin^{-1}\left(2\sqrt{\frac{2}{11}}\right) - \sin^{-1}\left(\frac{1}{\sqrt{55}}\right)\right)
\]

\[
\omega = 16\pi + 24\sin^{-1}\left(\frac{1}{5}\right) - 48\left(\sin^{-1}\left(\frac{8}{11}\right) - \sin^{-1}\left(\frac{1}{\sqrt{55}}\right)\right)
\]

Using formula: \(\sin^{-1}x - \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2})\) \(\forall \ |x|, |y| \leq 1\),

\[
\omega = 16\pi + 48\left(\frac{1}{2}\sin^{-1}\left(\frac{1}{5}\right)\right) - 48\left(\sin^{-1}\left(\frac{54}{11\sqrt{55}} - \frac{3}{11\sqrt{55}}\right)\right)
\]

Using formula: \(\frac{1}{2}\sin^{-1}x = \sin^{-1}\left(\sqrt{\frac{1-x^2}{2}}\right)\) \(\forall |x| \leq 1\),

\[
\omega = 16\pi + 48\left(\sin^{-1}\left(\sqrt{\frac{1 - \sqrt{5}}{2}}\right)\right) - 48\left(\sin^{-1}\left(\frac{12\sqrt{3} - \sqrt{3}}{11\sqrt{5}}\right)\right)
\]

\[
\omega = 16\pi + 48\sin^{-1}\left(\frac{1 - 2\sqrt{5}}{2}\right) - 48\left(\sin^{-1}\left(\frac{12\sqrt{3} - \sqrt{3}}{11\sqrt{5}}\right)\right)
\]

\[
\omega = 16\pi + 48\sin^{-1}\left(\frac{5 - 2\sqrt{5}}{10}\right) - 48\left(\sin^{-1}\left(\frac{11\sqrt{3}}{11\sqrt{5}}\right)\right)
\]

\[
\omega = 16\pi + 48\sin^{-1}\left(\frac{(\sqrt{3} - \sqrt{2})^2}{10}\right) - 48\left(\sin^{-1}\left(\frac{\sqrt{3}}{\sqrt{5}}\right)\right)
\]

\[
\omega = 16\pi + 48\sin^{-1}\left(\frac{\sqrt{3} - \sqrt{2}}{\sqrt{10}}\right) - 48\sin^{-1}\left(\frac{3}{5}\right)
\]

\[
\omega = 16\pi - 48\left(\sin^{-1}\left(\frac{3}{5}\right) - \sin^{-1}\left(\frac{\sqrt{3} - \sqrt{2}}{\sqrt{10}}\right)\right)
\]

\[
\omega = 16\pi - 48\left(\sin^{-1}\left(\frac{3}{5}\left(\sqrt{3} + \sqrt{2}\right)}{\sqrt{10}} - \frac{2}{5}\left(\sqrt{3} - \sqrt{2}\right)\right)\right)
\]

\[
\omega = 16\pi - 48\left(\sin^{-1}\left(\frac{3\sqrt{3} + \sqrt{2}}{5\sqrt{2}} - \frac{\sqrt{2}(\sqrt{3} - \sqrt{2})}{5\sqrt{2}}\right)\right)
\]

Applications of "HCR's Theory of Polygon" proposed by Mr H.C. Rajpoot (year 2014)
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Above result shows that the solid angle subtended by a truncated rhombic dodecahedron at its centre is $4\pi$ sr. It is true that the solid angle, subtended by any closed surface at any point inside it, is always $4\pi$ sr.

We know that a truncated rhombic dodecahedron has total 24 identical vertices at each of which two rectangular, one square & one equilateral triangular faces meet together. Thus we would analyse one of 24 identical vertices to compute solid angle subtended by a truncated rhombic dodecahedron at the same vertex.

**Solid angle subtended by truncated rhombic dodecahedron at any of its 24 identical vertices:**

Consider any of 24 identical vertices say vertex P of truncated rhombic dodecahedron. Join the end points A, B, C & D of the edges AP, BP, CP & DP, meeting at vertex P, to get a (plane) trapezium ABCD of sides $AB = s$, $BC = AD = s\sqrt{3}$ & $CD = 2s$ (as shown in fig-10).

Join the foot point Q of perpendicular PQ drawn from vertex P to the plane of trapezium ABCD, to the vertices A, B, C & D. Drop the perpendiculars MN & QE from midpoint M & foot point Q to the sides CD & BC respectively of trapezium ABCD (as shown in the fig-11)

We have found out that dihedral angle between square & equilateral triangular faces is $\angle MPN = \pi - \tan^{-1}\sqrt{2}$. The perpendicular PQ dropped from the vertex P to the plane of trapezium ABCD will fall at the point Q lying on the line MN (as shown in the fig-11).

In $\Delta MPN$ (see fig-12 below), using cosine formula as follows
\[ \cos \angle MPN = \frac{(PM)^2 + (PN)^2 - (MN)^2}{2(PM)(PN)} \]

\[ \cos(\pi - \tan^{-1} \sqrt{2}) = \frac{\left(\frac{s\sqrt{3}}{2}\right)^2 + s^2 - (MN)^2}{2\left(\frac{s\sqrt{3}}{2}\right)s} \Rightarrow \cos(\tan^{-1} \sqrt{2}) = \frac{\frac{3s^2}{4} + s^2 - MN^2}{s^2\sqrt{3}} \]

\[ \Rightarrow MN^2 = \frac{7s^2}{4} + s^2 \Rightarrow MN^2 = \frac{11s^2}{4} \Rightarrow MN = \frac{s\sqrt{11}}{2} \]

Now, the area of \(\triangle MPN\) (see fig-12), is given as follows

\[ \frac{1}{2}(MN)(PQ) = \frac{1}{2}(PM)(PN)\sin(\pi - \tan^{-1} \sqrt{2}) \]

\[ \left(\frac{s\sqrt{11}}{2}\right)(PQ) = \left(\frac{s\sqrt{3}}{2}\right)(s)\sin(\tan^{-1} \sqrt{2}) \]

\[ \sqrt{11}PQ = s\sqrt{3}\sin\left(\sin^{-1}\frac{2}{\sqrt{3}}\right) \Rightarrow PQ = \frac{s\sqrt{3}}{\sqrt{11}}\sqrt{\frac{2}{3}} = s\frac{2}{\sqrt{11}} \]

Using Pythagorean theorem in right \(\triangle PQM\) (see above fig-12 above), we get

\[ MQ = \sqrt{(PM)^2 - (PQ)^2} = \sqrt{\left(\frac{s\sqrt{3}}{2}\right)^2 - \left(\frac{2}{\sqrt{11}}\right)^2} = \sqrt{\frac{3s^2}{4} - \frac{2s^2}{11}} = \sqrt{\frac{25s^2}{44}} = \frac{5s}{2\sqrt{11}} \]

\[ \Rightarrow QN = MN - MQ = \frac{s\sqrt{11}}{2} - \frac{5s}{2\sqrt{11}} = \frac{3s}{\sqrt{11}} \]

Using Pythagorean theorem in right \(\triangle QMB\) (see above fig-11), we get

\[ BQ = \sqrt{(MQ)^2 + (MB)^2} = \sqrt{\left(\frac{5s}{2\sqrt{11}}\right)^2 + \left(\frac{s}{2}\right)^2} = \sqrt{\frac{25s^2}{44} + \frac{s^2}{4}} = \sqrt{\frac{36s^2}{44}} = \frac{3s}{\sqrt{11}} \]

Using Pythagorean theorem in right \(\triangle PQC\) (see above fig-13), we get

\[ QC = \sqrt{(PC)^2 - (PQ)^2} = \sqrt{\left(\sqrt{2}\right)^2 - \left(\frac{2}{\sqrt{11}}\right)^2} = \sqrt{2s^2 - \frac{2s^2}{11}} = \sqrt{\frac{20s^2}{11}} = 2s\frac{5}{\sqrt{11}} \]
In $\triangle QBC$ (see fig-11 above), using cosine formula as follows

$$\cos \alpha = \frac{(BQ)^2 + (BC)^2 - (QC)^2}{2(BQ)(BC)} = \frac{\left(\frac{3s}{\sqrt{11}}\right)^2 + \left(\frac{5s}{\sqrt{11}}\right)^2}{2 \left(\frac{3s}{\sqrt{11}}\right) \left(\frac{5s}{\sqrt{11}}\right)} = \frac{\frac{9s^2}{11} + \frac{3s^2}{2} - \frac{20s^2}{11}}{\frac{6s^2\sqrt{3}}{\sqrt{11}}}$$

$$\cos \alpha = \frac{9s^2 + 3s^2 - 20s^2}{6s^2\sqrt{3}} = \frac{22s^2}{6s^2\sqrt{3}} = \frac{\sqrt{11}}{3\sqrt{3}} = \frac{1}{3} \frac{11}{3}$$

In right $\triangle BEQ$ (see above fig-11),

$$\cos \alpha = \frac{BE}{BQ} \Rightarrow \frac{BE}{BQ} = \cos \alpha = \frac{3s}{\sqrt{11}} \cdot \frac{11}{3} \frac{1}{\sqrt{3}} = \frac{s}{\sqrt{3}}$$

$$\Rightarrow \quad \frac{EC}{BC} = \frac{BC - BE}{s\sqrt{3}} = \frac{s\sqrt{3} - \frac{s}{\sqrt{3}}}{s\sqrt{3}} = \frac{2s}{\sqrt{3}}$$

Using Pythagorean theorem in right $\triangle BEQ$ (see above fig-11), we get

$$QE = \sqrt{(BQ)^2 - (BE)^2} = \sqrt{\left(\frac{3s}{\sqrt{11}}\right)^2 - \left(\frac{s}{\sqrt{3}}\right)^2} = \sqrt{\frac{9s^2}{11} - \frac{s^2}{3}} = \sqrt{\frac{16s^2}{33}} = \frac{4s}{\sqrt{33}}$$

We know from HCR’s Theory of Polygon that the solid angle ($\omega$), subtended by a right triangle $OGH$ having perpendicular $p$ & base $b$ at any point $P$ at a normal distance $h$ on the vertical axis passing through the vertex $O$ (as shown in the fig-14), is given by HCR’s Standard Formula-1 as follows

$$\omega = \sin^{-1}\left(\frac{b}{\sqrt{b^2 + p^2}}\right) - \sin^{-1}\left(\frac{b}{\sqrt{b^2 + p^2}}\left(\frac{h}{\sqrt{h^2 + p^2}}\right)\right)$$

Now, the solid angle $\omega_{QMB}$ subtended by right $\triangle QMB$ at the vertex $P$ (see above fig-11) is obtained by substituting the corresponding values (as derived above) in above standard-1 formula i.e. base $b = MB = \frac{s}{2}$, perpendicular $p = MQ = \frac{5s}{2\sqrt{11}}$ & normal height $h = PQ = s\frac{2}{\sqrt{11}}$ as follows

$$\omega_{QMB} = \sin^{-1}\left(\frac{s}{2\sqrt{\frac{3s}{\sqrt{11}}}}\right) - \sin^{-1}\left(\frac{s}{2\sqrt{\frac{3s}{\sqrt{11}}}}\left(\frac{\frac{2}{\sqrt{11}}}{\frac{5s}{2\sqrt{11}}}\right)\right)$$

$$\omega_{QMB} = \sin^{-1}\left(\frac{\frac{2}{\sqrt{11}}}{\frac{3s}{\sqrt{11}}}\right) - \sin^{-1}\left(\frac{\frac{2}{\sqrt{11}}}{\frac{3s}{\sqrt{11}}}\left(\frac{\frac{2}{\sqrt{11}}}{\frac{5s}{2\sqrt{11}}}\right)\right) = \sin^{-1}\left(\frac{\sqrt{11}}{6}\right) - \sin^{-1}\left(\frac{\sqrt{11}}{6}\left(\frac{2}{\sqrt{33}}\right)\right)$$
\[ \omega_{\Delta QMB} = \sin^{-1} \left( \frac{\sqrt{11}}{6} \right) - \sin^{-1} \left( \frac{1}{2\sqrt{3}} \right) \]

Similarly, the solid angle \( \omega_{\Delta BEQ} \) subtended by right \( \Delta BEQ \) at the vertex \( P \) (see above fig-11) is obtained by substituting the corresponding values (as derived above) in above standard formula-1 i.e. base \( b = BE = \frac{s}{\sqrt{3}} \), perpendicular \( p = QE = \frac{4s}{\sqrt{33}} \) & normal height \( h = PQ = s \sqrt{\frac{2}{11}} \) as follows

\[
\omega_{\Delta BEQ} = \sin^{-1} \left( \frac{\sqrt{3}s}{\sqrt{33}} \right) - \sin^{-1} \left( \frac{\sqrt{3}s}{s \sqrt{11}} \right) = \sin^{-1} \left( \frac{1}{3} \sqrt{\frac{11}{3}} \right) - \sin^{-1} \left( \frac{1}{3} \sqrt{\frac{11}{3}} \right)
\]

\[ \omega_{\Delta BEQ} = \sin^{-1} \left( \frac{1}{3} \sqrt{\frac{11}{3}} \right) - \sin^{-1} \left( \frac{1}{3} \sqrt{\frac{11}{3}} \right) \]

Similarly, the solid angle \( \omega_{\Delta ACE} \) subtended by right \( \Delta ACE \) at the vertex \( P \) (see above fig-11) is obtained by substituting the corresponding values (as derived above) in above standard formula-1 i.e. base \( b = EC = \frac{2s}{\sqrt{3}} \), perpendicular \( p = QE = \frac{4s}{\sqrt{33}} \) & normal height \( h = PQ = s \sqrt{\frac{2}{11}} \) as follows

\[
\omega_{\Delta ACE} = \sin^{-1} \left( \frac{2s}{\sqrt{3}} \right) - \sin^{-1} \left( \frac{2s}{\sqrt{3}} \right) = \sin^{-1} \left( \frac{11}{15} \right) - \sin^{-1} \left( \frac{11}{15} \right)
\]

\[ \omega_{\Delta ACE} = \sin^{-1} \left( \frac{11}{15} \right) - \sin^{-1} \left( \frac{1}{\sqrt{5}} \right) \]

Similarly, the solid angle \( \omega_{\Delta QNC} \) subtended by right \( \Delta QNC \) at the vertex \( P \) (see above fig-11) is obtained by substituting the corresponding values (as derived above) in above standard formula-1 i.e. base \( b = NC = s \), perpendicular \( p = QN = \frac{3s}{\sqrt{11}} \) & normal height \( h = PQ = s \sqrt{\frac{2}{11}} \) as follows
Now, according to HCR’s Theory of Polygon, the solid angle $\omega_{MBCN}$ subtended by the trapezium MBCN at the vertex P (see above fig-11) is the algebraic sum of solid angles subtended by the right triangles $\Delta QMB, \Delta BEQ, \Delta CEQ$ & $\Delta QNC$ which is given as follows

$$\omega_{MBCN} = \omega_{\Delta QMB} + \omega_{\Delta BEQ} + \omega_{\Delta CEQ} + \omega_{\Delta QNC}$$

Substituting the corresponding values of solid angles (derived above) as follows

$$\omega_{MBCN} = \left( \sin^{-1}\left(\frac{\sqrt{11}}{6}\right) - \sin^{-1}\left(\frac{2}{3}\sqrt{3}\right) \right) + \left( \sin^{-1}\left(\frac{11}{3}\sqrt{3}\right) - \sin^{-1}\left(\frac{1}{3}\right) \right)$$

$$+ \left( \sin^{-1}\left(\frac{11}{15}\sqrt{5}\right) - \sin^{-1}\left(\frac{1}{\sqrt{5}}\right) \right) + \left( \sin^{-1}\left(\frac{11}{5}\sqrt{5}\right) - \sin^{-1}\left(\frac{1}{\sqrt{10}}\right) \right)$$

$$= \sin^{-1}\left(\frac{\sqrt{11}}{6}\right) + \sin^{-1}\left(\frac{11}{3}\sqrt{3}\right) - \sin^{-1}\left(\frac{2}{3}\sqrt{3}\right) + \sin^{-1}\left(\frac{11}{15}\sqrt{5}\right) + \sin^{-1}\left(\frac{11}{5}\sqrt{5}\right)$$

Using formula: $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) \quad \forall \ |x|, |y| \leq 1$

$$= \left( \sin^{-1}\left(\frac{\sqrt{11}}{6} + \frac{5}{3}\sqrt{\frac{11}{3}}\right) \right) - \left( \sin^{-1}\left(\frac{2}{3}\sqrt{3} + \frac{5}{3}\sqrt{\frac{1}{3}}\right) \right)$$

$$+ \left( \sin^{-1}\left(\frac{11}{15}\sqrt{5} + \frac{2}{\sqrt{5}}\sqrt{\frac{11}{5}}\right) \right) - \left( \sin^{-1}\left(\frac{1}{\sqrt{5}} + \frac{2}{\sqrt{10}}\sqrt{\frac{1}{10}}\right) \right)$$

$$= \sin^{-1}\left(\frac{4\sqrt{11}}{18\sqrt{3}} + \frac{5\sqrt{11}}{18\sqrt{3}}\right) + \sin^{-1}\left(\frac{4}{9\sqrt{3}} + \frac{5}{9\sqrt{3}}\right) + \left( \sin^{-1}\left(\frac{3\sqrt{11}}{10\sqrt{3}} + \frac{2\sqrt{11}}{10\sqrt{3}}\right) \right) - \sin^{-1}\left(\frac{3}{5\sqrt{2}} + \frac{2}{5\sqrt{2}}\right)$$

$$= \sin^{-1}\left(\frac{9\sqrt{11}}{18\sqrt{3}} - \sin^{-1}\left(\frac{9}{9\sqrt{3}}\right) + \left( \pi - \sin^{-1}\left(\frac{5\sqrt{11}}{10\sqrt{3}}\right) \right) - \sin^{-1}\left(\frac{5}{5\sqrt{2}}\right)\right)$$
\[
\sin^{-1}\left(\frac{1}{2} \sqrt{\frac{11}{3}}\right) - \sin^{-1}\left(\frac{1}{\sqrt{3}}\right) + \pi - \sin^{-1}\left(\frac{1}{2} \sqrt{\frac{11}{3}}\right) - \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)
\]
\[
= \sin^{-1}\left(\frac{1}{2} \sqrt{\frac{11}{3}}\right) - \sin^{-1}\left(\frac{1}{\sqrt{3}}\right) + \pi - \sin^{-1}\left(\frac{1}{2} \sqrt{\frac{11}{3}}\right) - \frac{\pi}{4}
\]
\[
= \frac{3\pi}{4} - \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)
\]
\Rightarrow \omega_{MBCN} = \frac{3\pi}{4} - \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)

Thus, using symmetry in trapezium ABCD (see above fig-11), the solid angle \(\omega_{ABCD}\) subtended by the trapezium ABCD at the vertex P of truncated rhombic dodecahedron will be twice the solid angle \(\omega_{MBCN}\) subtended by the trapezium MBCN at the vertex P, as follows

\[
\omega_{ABCD} = 2\omega_{MBCN} = 2\left(\frac{3\pi}{4} - \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)\right) = \frac{3\pi}{2} - 2\sin^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{3\pi}{2} - \sin^{-1}\left(2 \cdot \frac{1}{\sqrt{3}} \cdot \frac{2}{\sqrt{5}}\right)
\]
\[
= \frac{3\pi}{2} - \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \pi + \left(\frac{\pi}{2} - \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right) = \pi + \cos^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \pi + \sin^{-1}\left(\frac{1}{3}\right)
\]

It’s worth noticing that the solid angle \(\omega_{P}\) subtended by truncated rhombic dodecahedron at its vertex P will be equal to the solid angle \(\omega_{ABCD}\) subtended by the trapezium ABCD at the vertex P.

Hence, the solid angles \(\omega_{P}\) subtended by a truncated rhombic dodecahedron at any of its 24 identical vertices (at each of which two rectangular, one square & one regular triangular faces meet), is given as follows

\[
\omega_{P} = \pi + \sin^{-1}\left(\frac{1}{3}\right) \text{ sr} \approx 3.481429563 \text{ sr} \tag{11}
\]

Paper model of a truncated rhombic dodecahedron: In order to make the paper model of a truncated rhombic dodecahedron having 12 congruent rectangular faces, 6 congruent square faces & 8 congruent equilateral triangular faces, it first requires the net of 26 faces to be drawn on a paper sheet

1: Prepare a net of 26 faces out of which there are 12 congruent rectangular faces each with length \(s\sqrt{2}\) & width \(s\), 6 congruent square faces each with side \(s\sqrt{2}\) & 8 congruent equilateral triangular faces each with side \(s\) on the plain sheet of paper (as shown by left image in fig-15)

2: Fold each of 26 faces about its common (junction) edge such that the open edges of faces overlap one another & thus the net conforms to a closed surface. Glue the faces at the coincident edges to retain the shape of a truncated rhombic dodecahedron. (as shown by right image in fig-15)

Figure 15: A net (left) of 12 congruent rectangular, 6 congruent square & 8 congruent equilateral triangular faces, is folded to conform to the shape of a truncated rhombic dodecahedron (right).
**Summary:** Let there be a truncated rhombic dodecahedron having 12 congruent rectangular faces each with length $\sqrt[3]{2}$ and width $s$, 6 congruent square faces each with side $s\sqrt[3]{2}$, and 8 congruent equilateral triangular faces each with side $s$, 48 edges and 24 identical vertices then all its important parameters are determined as tabulated below:

<table>
<thead>
<tr>
<th>Parameter Description</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius ($R$) of circumscribed sphere passing through all 24 vertices</td>
<td>$s\sqrt[3]{3} \approx 1.732050808 s$</td>
</tr>
<tr>
<td>Normal distances $H_R$, $H_S$ &amp; $H_T$ of rectangular, square &amp; regular triangular faces from the centre of truncated rhombic dodecahedron</td>
<td>$H_R = \frac{3s}{2}$, $H_S = s\sqrt[3]{2} \approx 1.414213562 s$ &amp; $H_T = 2s\sqrt[3]{\frac{2}{3}} \approx 1.632993162 s$</td>
</tr>
<tr>
<td>Surface area ($A_s$)</td>
<td>$A_s = 2s^2(6\sqrt[3]{2} + 6 + \sqrt[3]{3}) \approx 32.43466436 s^2$</td>
</tr>
<tr>
<td>Volume ($V$)</td>
<td>$V = \frac{34s^3\sqrt[3]{2}}{3} \approx 16.02775371s^3$</td>
</tr>
<tr>
<td>Mean radius ($R_m$) or radius of sphere having volume equal to that of the truncated rhombic dodecahedron</td>
<td>$R_m = s\left(\frac{17}{\pi\sqrt[3]{2}}\right)^{1/3} \approx 1.564088599 s$</td>
</tr>
<tr>
<td>Dihedral angle $\theta_{RS}$ between any two adjacent rectangular &amp; square faces</td>
<td>$\theta_{RS} = \frac{3\pi}{4} = 135^\circ$</td>
</tr>
<tr>
<td>Dihedral angle $\theta_{RT}$ between any two adjacent rectangular &amp; square faces</td>
<td>$\theta_{RT} = \pi - \tan^{-1}\frac{1}{\sqrt[3]{2}} \approx 144^\circ 44' 8.2''$</td>
</tr>
<tr>
<td>Dihedral angle $\theta_{ST}$ between square &amp; equilateral triangular faces meeting at the same vertex</td>
<td>$\theta_{ST} = \pi - \tan^{-1}\sqrt[3]{2} \approx 125^\circ 15' 51.8''$</td>
</tr>
<tr>
<td>Dihedral angle $\theta_{RR}$ between any two rectangular faces meeting at the same vertex</td>
<td>$\theta_{RR} = \frac{2\pi}{3} = 120^\circ$</td>
</tr>
<tr>
<td>Solid angles $\omega_R$, $\omega_S$ &amp; $\omega_T$ subtended by rectangular, square &amp; equilateral triangular faces at the centre of truncated rhombic dodecahedron</td>
<td>$\omega_R = 4\sin^{-1}\left(\frac{1}{\sqrt[3]{55}}\right) sr \approx 0.541 sr$, $\omega_S = 4\sin^{-1}(0.2) sr \approx 0.805 sr$ $\omega_T = 2\pi - 6\sin^{-1}\left(\frac{2}{\sqrt[3]{11}}\right) sr \approx 0.155210814 sr$</td>
</tr>
<tr>
<td>Solid angle $\omega_V$ subtended by truncated rhombic dodecahedron at its vertex</td>
<td>$\omega_V = \pi + \sin^{-1}\left(\frac{1}{3}\right) sr \approx 3.481429563 sr$</td>
</tr>
</tbody>
</table>

**Note:** Above articles had been derived & illustrated by Mr H.C. Rajpoot (M Tech, Production Engineering)  
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